# Second-order hyperbolic Fuchsian systems. General theory

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**Abstract**

We introduce a class of singular partial differential equations, the *second-order hyperbolic Fuchsian systems,* and we investigate the associated initial value problem when data are imposed on the singularity. First of all, we analyze a class of equations in which hyperbolicity is not assumed and we construct asymptotic solutions of arbitrary order. Second, for the proposed class of second-order hyperbolic Fuchsian systems, we establish the existence of solutions with prescribed asymptotic behavior on the singularity. Our proof is based on a new scheme which is also suitable to design numerical approximations. Furthermore, as shown in a follow-up paper, the second-order Fuchsian framework is appropriate to handle Einstein's field equations for Gowdy symmetric spacetimes and allows us to recover (and slightly generalize) earlier results by Rendall and collaborators, while providing a direct approach leading to accurate numerical solutions. The proposed framework is also robust enough to encompass matter models arising in general relativity.

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# 1 Introduction

This is the first of a series of papers [7, 8] devoted to solving the initial value problem for certain classes of spacetimes of general relativity. Specifically, we are interested in spacetimes enjoying certain symmetries, especially the Gowdy symmetry, and in a formulation when data are imposed on a singular hypersurface where curvature generically blows-up.

For instance, one may consider $(3+1)$-dimensional, vacuum spacetimes $(M, g)$ with spatial topology $T^3$, satisfying the vacuum Einstein equations under the Gowdy symmetry assumption, i.e. under the existence of an Abelian $T^2$ isometry group with spacelike orbits whose so-called "twist constants" [9] vanish identically. These so-called Gowdy spacetimes on $T^3$ were first studied in [12]. A combination of theoretical and numerical works has led to a detailed picture of the behavior of solutions to the Einstein equations as one approaches the singular boundary of such spacetimes; see [9, 13, 18, 21, 22, 23].

For the above analysis, one important tool was provided by Rendall and his collaborators [2, 14, 19] who developed the so-called *Fuchsian method* to handle the singular evolution equations associated with the Einstein equations for such spacetimes. This method allowed the authors to derive precise information about the behavior of solutions near the singularity, which was a key step in the general proof of Penrose's strong cosmic conjecture eventually established by Ringström [23].

Our aim here and in the follow-up papers [7, 8] is two-fold. On one hand, we re-visit Rendall's theory (which covers smooth solutions to first-order equations) and we develop here a well-posedness theory in Sobolev spaces for the class of *second-order hyperbolic Fuchsian systems*, defined below in Section 3. As we will show, this class includes many systems of equations arising in general relativity.

On the other hand, following a strategy initially proposed in Amorim, Bernardi, and LeFloch [1], we are interested in the numerical approximation of these Fuchsian equations when data are imposed on the singularity of the spacetime and one evolves the solution *from* the singularity. In contrast, standard numerical approaches consider the evolution *toward* the singularity. Our main improvement here upon [1] is that, in short, no restriction need be imposed on the coefficients of the Fuchsian system, provided asymptotic expansions of arbitrary large order are sought. This issue will be developed in [7].

The present paper is theoretical in nature, and our main contributions can be summarized as follows:

- **Second-order formulation.** First-order Fuchsian systems have been used successfully to handle the equations describing (vacuum) Gowdy spacetimes [2, 14, 19]. However, we argue here that second-order hyperbolic Fuchsian systems, as we define them in this paper, arise more naturally in the applications. For instance, Gowdy spacetimes described by second-order partial differential equations, and it is natural to keep the second-order structure. In particular, expansions required in the theoretical analysis (as well as in actual computations required for the numerical discretization) are also more natural with the second-order formulation.

- **Hyperbolicity property.** In addition, in the applications to general relativity, ensuring and checking the hyperbolicity of the equations under consideration (after a suitable reduction of the original equations) is expected to be more convenient with the second-order

formulation. For instance, it is easier to recognize from the original second-order system if the equations form a system of coupled wave equations, while this property is much less evident in the first-order formulation. As the discussion in [19] demonstrates, it can be cumbersome to formulate a system of coupled non-linear wave equations as a first-order hyperbolic Fuchsian system satisfying all the properties required for local well-posedness.

- **Singular part.** We also introduce here a construction algorithm which includes the singular part of the solution. This is different from the classical first-order Fuchsian approach, where one first makes an ansatz by removing from the solution its (expected) singular part, and then applies the Fuchsian theory to the (regular) remainder. Our approach leads to the notion of the singular initial value problem for Fuchsian hyperbolic equations which can be understood as a generalization of the initial value problem for (standard) hyperbolic equations. This singular initial value problem covers cases whose singularity is oscillatory in nature, in a manner consistent with the so-called BKL conjecture (introduced by Belinsky, Khalatnikov, and Lifshitz).

We shall, first, define the class of systems of interest, that is, the second-order (hyperbolic) Fuchsian systems and, then, investigate the associated initial value problem when data are imposed on the singularity. Precisely, in Section 2, we analyze a class of equations in which hyperbolicity is not assumed and we construct asymptotic solutions of arbitrary order. In Section 3, we treat the proposed class of second-order hyperbolic Fuchsian systems, and establish the existence of solutions with prescribed asymptotic behavior on the singularity.

In [7], we will apply our Fuchsian framework and treat the class of Gowdy spacetimes, and this will allow us to recover (and slightly generalize) earlier results by Rendall and collaborators, while providing a more direct approach. Although conceptually similar, the proposed analysis based on second-order equations lead to a simpler description and provides a definite advantage for the applications. Moreover, as we will demonstrate in [7]. the approach introduced in the present paper can be cast into a discretization scheme and allows us to numerically and accurately compute solutions to the initial value problem. Our theory also turns out to be robust enough to extend to matter models, especially to the Einstein-Euler equations [3, 15, 16, 17], as discussed in [8].

# 2 Second-order Fuchsian systems

## 2.1 Terminology and objectives

In this section, we rely mainly on techniques for ordinary differential equations (ODE's). The main theory of interest developed in the next section (Section 3) will require an hyperbolicity assumption which is not yet made at this stage. The main purpose of this section is to present some important terminology and concepts within the simple framework of ODE's, and to point out some difficulty arising with singular equations.

First of all, $t \geq 0$ denoting the time variable, the operator

$$D := t\partial_t$$

will often be used, rather than the partial derivative $\partial_t$. Indeed, the weight $t$ is convenient to handle asymptotic expansions near the singularity $t = 0$. Occasionally, we write $D_t$ instead of $D$, especially when several time variables are involved.

**Definition 2.1** (Second-order Fuchsian systems)**.** *A second-order Fuchsian system is a system of partial differential equations of the form*

$$D^2u(t,x) + 2A(x)\,Du(t,x) + B(x)\,u(t,x) = f[u](t,x) \tag{2.1}$$

*with unknown function* $u : (0,\delta] \times U \to \mathbb{R}^n$ *(for some* $\delta > 0$ *and interval* $U \subset \mathbb{R}$*), where the coefficients* $A = A(x)$ *and* $B = B(x)$ *are diagonal,* $n \times n$ *matrix-valued maps defined on* $U$*, and the source-term* $f = f[u](t,x)$ *is an* $n$*-vector-valued map of the form*

$$f[u](t,x) := f(t,x,u,Du,\partial_x u,\partial_x Du,\ldots,\partial_x^k u,\partial_x^k Du), \tag{2.2}$$

*for some integer* $k \geq 0$*.*

We assume that the coefficients $A$ and $B$ do not depend on $t$ and we derive our results under this assumption. The generalization to time-dependent coefficients does not bring essential difficulties. The assumption that the matrices $A$, $B$ are diagonal is not a genuine restriction, if the system with arbitrary matrices $A$, $B$ can be recast in diagonal form, i.e. if $A$, $B$ admit a common basis of eigenvectors. Under this condition (which is always satisfied in the applications of interest in general relativity), the system is "essentially decoupled" since the coupling takes place in terms of non-leading order, only.

We denote the eigenvalues of $A$ and $B$ by $a^{(1)},\ldots,a^{(n)}$ and $b^{(1)},\ldots,b^{(n)}$, respectively. When it is not necessary to specify the superscripts, we just write $a, b$ to denote any eigenvalues of $A, B$. With this convention, we introduce:

$$\lambda_1 := a + \sqrt{a^2 - b}, \quad \lambda_2 := a - \sqrt{a^2 - b}. \tag{2.3}$$

It will turn out that these coefficients describe the expected behavior at $t = 0$ of general solutions to (2.1).

In Definition 2.1, the assumption that $U$ is a one-dimensional domain makes the presentation simpler, but most results below remain valid for arbitrary spatial dimensions. For definiteness and without much loss of generality, we assume throughout this paper that all functions under consideration are periodic in the spatial variable $x$ and that $U$ is the periodicity domain. All data and solutions are extended by periodicity outside the interval $U$.

The left-hand side of (2.1) is referred to as the principal part of the system. The reason for incorporating certain lower derivative terms in the principal part is that we expect these terms to be of the same leading-order at the singularity $t = 0$. In contrast, the source-term is anticipated as negligible in some sense there, see below. Observe that, at this level of generality, there is some freedom in bringing terms from the principal part to the right-hand side, and absorbing them into the source-function $f$ (or vice-versa). This freedom has several (interesting) consequences, as we will discuss later on: roughly speaking, some normalization will be necessary later, yet at this stage, we do not fix the behavior of $f$ at $t = 0$.

We are mainly interested in solving a *singular* initial value problem associated with (2.1), with data prescribed on the singularity $t = 0$, in a sense made precise later on. The fundamental question is, of course, to determine conditions on the data and coefficients ensuring existence and uniqueness of a solution $u$. It will turn out that the behavior at $t = 0$ cannot be prescribed arbitrarily, but is tight to the value of the coefficients $\lambda_1$ and $\lambda_2$ defined in (2.3). Indeed, we will specify the behavior of solutions at $t = 0$, in terms of freely specifiable functions (the data on the singularity), and derive an asymptotic expression of arbitrary order providing the asymptotic form of general solutions.

## 2.2 The case of linear ODE's depending on $x$ as a parameter

### Explicit formula

We begin our investigation of the second-order Fuchsian systems (2.1) by treating the case $f = w(t,x)$ for some given function $w$. We are led to consider a family of scalar ordinary differential equations that are completely independent from each other; recall that the matrices $A$ and $B$ of the principal part are diagonal. Without loss of generality, we thus assume that $n = 1$ throughout the present section. In turn, the spatial variable $x$ is treated as a *parameter* which we do not need to write explicitly yet. As we will see, it is instructive to express the general solution in this elementary case, as we now do.

Consider the following singular, inhomogeneous, singular ordinary differential equation

$$D^2u(t) + 2a\,Du(t) + b\,u(t) = w(t) \tag{2.4}$$

with unknown $u = u(t)$, where $w = w(t)$ is a given locally integrable function. Further integrability of $w$ near $t = 0$ will be imposed shortly below. Recall that $a$ and $b$ are constant in $t$ and, $\lambda_1, \lambda_2$ were defined in (2.3). We begin with a formal result and the convergence of the integral terms will be discussed rigorously later.

**Proposition 2.2** (Linear second-order Fuchsian ODE. Formal version). *General solutions of the inhomogeneous singular ordinary differential equation* (2.4) *are given by*

$$u(t) = \begin{cases} u_*\,t^{-a}\,\ln t + u_{**}\,t^{-a} + \displaystyle\int_1^\infty w(t/\zeta)\zeta^{-a-1}\ln\zeta\,d\zeta, & a^2 = b, \\ u_*\,t^{-\lambda_1} + u_{**}\,t^{-\lambda_2} & \\ \quad + \frac{1}{\lambda_1-\lambda_2}\displaystyle\int_1^\infty w(t/\zeta)\left(\zeta^{-\lambda_2-1} - \zeta^{-\lambda_1-1}\right)d\zeta, & a^2 \neq b, \end{cases} \tag{2.5}$$

*in which $u_*$ and $u_{**}$ are prescribed data. Alternatively, one can write* (2.5) *in the form*

$$u(t) = \begin{cases} u_*\,t^{-a}\,\ln t + u_{**}\,t^{-a} + t^{-a}\displaystyle\int_0^t w(s)s^{a-1}\ln\frac{t}{s}\,ds, & a^2 = b, \\ u_*\,t^{-\lambda_1} + u_{**}\,t^{-\lambda_2} & \\ \quad + \frac{1}{\lambda_1-\lambda_2}\left(t^{-\lambda_2}\displaystyle\int_0^t w(s)s^{\lambda_2-1}ds - t^{-\lambda_1}\int_0^t w(s)s^{\lambda_1-1}\right), & a^2 \neq b. \end{cases}$$

*Proof.* In the rescaled time variable $\eta := -\ln t$, equation (2.4) has constant coefficients, indeed

$$\widehat{u}''(\eta) - 2a\,\widehat{u}'(\eta) + b\,\widehat{u}(\eta) = \widehat{w}(\eta), \tag{2.6}$$

where $\widehat{u}(\eta) := u(e^{-\eta})$ and $\widehat{w}(\eta) := w(e^{-\eta})$ and the prime $'$ denotes a derivative with respect to $\eta$. This is nothing but a linear harmonic oscillator equation with friction term $-2a\,\widehat{u}'$ and forcing term $\widehat{w}$. The singularity of (2.4) at $t = 0$ corresponds to the singularity at infinity $\eta = \infty$.

First, we seek for general solutions of the homogeneous equation for $\widehat{w} \equiv 0$. From the ansatz $\widehat{u} = e^{\lambda\eta}$ we obviously get the roots defined in (2.3). The general solution of the homogeneous equation is thus

$$\widehat{u}(\eta) = \begin{cases} -u_*\,\eta\,e^{a\eta} + u_{**}\,e^{a\eta}, & a^2 = b, \\ u_*\,e^{\lambda_1\eta} + u_{**}\,e^{\lambda_2\eta}, & a^2 \neq b, \end{cases}$$

where $u_*$ and $u_{**}$ are constants with respect to $\eta$ (and the negative sign is chosen for convenience in the following discussion).

Particular solutions of the general inhomogeneous equation (2.6) are easily constructed by the Duhamel principle. Let $\widetilde{u} := \widetilde{u}(\eta)$ be the solution of the homogeneous equation for vanishing data $\widetilde{u}(0) = 0$ and $\widetilde{u}'(0) = -1$. For any given $\widehat{w}$, the function

$$\widehat{u}(\eta) = \int_{-\infty}^{0} \widetilde{u}(\tau)\, \widehat{w}(\eta - \tau)\, d\tau$$

is a particular solution of the inhomogeneous equation. (This is true only formally at this stage, since we have not yet checked under which conditions the integral exists and can be differentiated.) Hence, the general solution $u$ of (2.6) reads

$$\widehat{u}(\eta) = \begin{cases} -u_*\, \eta\, e^{a\eta} + u_{**}\, e^{a\eta} - \int_{-\infty}^{0} \tau e^{a\tau} \widehat{w}(\eta - \tau) d\tau, & a^2 = b, \\ u_*\, e^{\lambda_1 \eta} + u_{**}\, e^{\lambda_2 \eta} & \\ \quad -\frac{1}{\lambda_1 - \lambda_2} \int_{-\infty}^{0} \left(e^{\lambda_1 \tau} - e^{\lambda_2 \tau}\right) \widehat{w}(\eta - \tau) d\tau, & a^2 \neq b. \end{cases}$$

Returning to the original time $t > 0$, we find

$$u(t) = \begin{cases} u_*\, t^{-a} \ln t + u_{**}\, t^{-a} - \int_{\infty}^{1} (-\ln \zeta) \zeta^{-a} \widehat{w}(\ln(\zeta/t))(-1/\zeta) d\zeta, & a^2 = b, \\ u_*\, t^{-\lambda_1} + u_{**}\, t^{-\lambda_2} & \\ \qquad -\frac{1}{\lambda_1 - \lambda_2} \int_{\infty}^{1} \left(\zeta^{-\lambda_1} - \zeta^{-\lambda_2}\right) \widehat{w}(\ln(\zeta/t))(-1/\zeta) d\zeta, & a^2 \neq b, \end{cases}$$

and by substituting $\tau = -\ln \zeta$, this concludes the proof. □

**The spatial coordinate $x$ as a parameter**

For the later discussion, it is convenient to write the spatial variable $x$ explicitly as a parameter now. Define $\Gamma(x) := \sqrt{a(x)^2 - b(x)}$ which might be real or imaginary dependent on the values of the coefficients. If there are points $x_0 \in U$ so that $\Gamma(x_0) = 0$ and other points $x_1 \in U$ with $\Gamma(x_1) \neq 0$, then we will renormalize the coefficients $u_*(x)$ and $u_{**}(x)$ in (2.5) as follows. In order to obtain a continuous transition from the non-degenerate case $\Gamma \neq 0$ to the degenerate case $\Gamma = 0$, let us first rename the coefficient functions for the case $a^2 \neq b$ in (2.5) to $\hat{u}_*$ and $\hat{u}_{**}$. Now if we set

$$\hat{u}_*(x) = \frac{u_*(x) - u_{**}(x)/\Gamma(x)}{2}, \quad \hat{u}_{**}(x) = \frac{u_*(x) + u_{**}(x)/\Gamma(x)}{2}, \tag{2.7}$$

and choose $u_*(x)$, $u_{**}(x)$ to be, say, continuous functions, then the function determined by the two leading terms in (2.11) is continuous in $x$ for all $t > 0$ even at $x = x_0$, provided $\Gamma$ is continuous. Indeed the full general solution $u(t,x)$ of (2.4) is continuous in $x$ for all $t > 0$ in this case.

In view of Proposition 2.2 it is natural to define the solution operator $H$ associated with a source function $w = w(t,x)$ by

$$(H[w])(t,x) := \begin{cases} t^{-a(x)} \displaystyle\int_0^t w(s,x) s^{a(x)-1} \ln \frac{t}{s}\, ds, & (a(x))^2 = b(x), \\ \dfrac{1}{\lambda_1(x) - \lambda_2(x)} \Big( t^{-\lambda_2(x)} \displaystyle\int_0^t w(s) s^{\lambda_2(x)-1} ds & \\ \qquad\qquad -t^{-\lambda_1(x)} \displaystyle\int_0^t w(s,x) s^{\lambda_1(x)-1} \Big), & (a(x))^2 \neq b(x). \end{cases} \tag{2.8}$$

It represents the solution of (2.4) for the choice $u_* = u_{**} = 0$, at least on the formal level so far. According to the previous discussion, the non-degenerate case $a^2 \neq b$ in the definition converges

to the degenerate case $a^2 = b$ at degenerate points continuously, provided the coefficients are continuous, and vice versa.

Fixing some $\delta > 0$, we now assume that the source $w$ in (2.4) belongs to $C^{l\times m}((0,\delta]\times U)$, that is, $w$ is $l$-times continuously differentiable with respect $t$ and $m$-times continuously differentiable with respect to $x$ on $(0,\delta] \times U \subset \mathbb{R}^2$. Here, $l$ and $m$ are non-negative integers. Moreover, we assume that the coefficients $a$ and $b$ of the equation are $C^m(U)$. In this case, the general theory of ordinary differential equations implies that the solution $u(t,x)$ of (2.4) depends as a $C^m$ function on $x$. Hence, if $H[w]$ rigorously represents a particular solution of (2.4), then i) the function $H[w](t,x)$ is in $C^m(U)$ with respect to $x$ for each $t > 0$, ii) we can take the spatial derivatives under the integral, iii) each spatial derivative of $H[w](t,x)$ converges from the non-degenerate to the degenerate case at degenerate points as a continuous function, and vice versa.

### Behavior near the singular time

Now we go beyond a formal derivation and determine precise conditions on $w$ under which the integrals in (2.5) make sense and (2.5) provides actual solutions of (2.4). We use here the notation $\Re$ for the real part of a complex number.

**Proposition 2.3** (Pointwise properties of the solution operator $H$)**.** *Fix some $\delta > 0$, a compact set $K \subset U$, and $l, m \geq 0$, and let $w$ be a function in $C^{l\times m}((0,\delta] \times U)$, and $a, b \in C^m(U)$. In addition, suppose that $w$ satisfies the following asymptotic conditions: there exists a constant $\alpha$ such that*

$$\alpha > -\Re\lambda_2(x), \qquad x \in K$$

*and, for all $0 \leq p \leq l$, $0 \leq q \leq m$,*

$$\sup_K \left|D^p \partial_x^q w(t,\cdot)\right| = O(t^\alpha).$$

*Then, the operator $H$ given by (2.8) is well-defined and, if $l \geq 2$, it provides a particular classical solution of (2.4). Moreover, we have $H[w] \in C^{l\times m}((0,\delta] \times K)$ with, for all sufficiently small $\epsilon > 0$,*

$$\sup_K \left|D^p \partial_x^q H[w](t,\cdot)\right| = O(t^{\alpha-\epsilon}).$$

*In addition, for $1 \leq p \leq l$ one has*

$$(D_t^p H[w])(t,x) = \int_0^t D_s^{p-1} w(s,x) s^{a(x)-1} \left(a(x) \ln \frac{s}{t} + 1\right) ds \tag{2.9}$$

*when $a^2(x) = b(x)$ and, otherwise,*

$$\begin{aligned}(D_t^p H[w])(t,x) = \frac{1}{\lambda_1(x) - \lambda_2(x)} \Bigg( &- \lambda_2(x) t^{-\lambda_2(x)} \int_0^t D_s^{p-1} w(s,x) s^{\lambda_2(x)-1} ds \\ &+ \lambda_1(x) t^{-\lambda_1(x)} \int_0^t D_s^{p-1} w(s,x) s^{\lambda_1(x)-1} ds \Bigg).\end{aligned} \tag{2.10}$$

We note that we are allowed to choose $\epsilon = 0$ in the previous proposition only if $a(x)$ and $b(x)$ are constants in space or for $q = 0$. The constant $\epsilon > 0$ is necessary in order to control logarithms which arise when spatial derivatives are taken of functions involving spatially dependent powers of $t$.

*Proof.* The proof is quite direct and we only show the derivation of the formulas (2.9) and (2.10). For definiteness, we treat the case $a^2(x) \neq b(x)$ only, since the other case is treated similarly and the transition between the two cases can be obtained by appropriate limiting procedures. Consider the expression for $H$ in (2.5). Under our regularity assumptions, all derivatives are calculated by differentiation under the integral sign, and we obtain

$$(D^p H[w])(t,x) = \frac{1}{\lambda_1 - \lambda_2} \int_1^\infty D_t^p w(t/\zeta, x) \left(\zeta^{-\lambda_2 - 1} - \zeta^{-\lambda_1 - 1}\right) d\zeta.$$

As done earlier, we introduce the new variable $s := t/\zeta$ and observe that $D_t = t\partial_t = s\partial_s =: D_s$ for fixed $\zeta$. Hence, we obtain

$$\begin{aligned}
(D^p H[w])(t,x) &= \frac{1}{\lambda_1 - \lambda_2} \int_0^t D_s^p w(s,x) \left(\zeta^{-\lambda_2 - 1} - \zeta^{-\lambda_1 - 1}\right) \frac{\zeta}{s} ds \\
=& \frac{1}{\lambda_1 - \lambda_2} \int_0^t \partial_s (D_s^{p-1} w)(s,x) \left(t^{-\lambda_2} s^{\lambda_2} - t^{-\lambda_1} s^{\lambda_1}\right) ds \\
=& \frac{1}{\lambda_1 - \lambda_2} \Bigg( t^{-\lambda_2} \left( D_s^{p-1} w(s,x) s^{\lambda_2} \Big|_0^t - \lambda_2 \int_0^t D_s^{p-1} w(s,x) s^{\lambda_2 - 1} ds \right) \\
&\quad - t^{-\lambda_1} \left( D_s^{p-1} w(s,x) s^{\lambda_1} \Big|_0^t - \lambda_1 \int_0^t D_s^{p-1} w(s,x) s^{\lambda_1 - 1} ds \right) \Bigg),
\end{aligned}$$

where we used integration by parts. Now, in view of our regularity assumptions, we conclude that all terms here have a limit when $t \to 0$. All terms, except for the main integrals, either vanish or cancel. □

## 2.3 Asymptotic solutions of second-order Fuchsian systems

### Asymptotic data and canonical expansions

We now return to the non-linear problem (2.1). We are going to identify the "canonical" asymptotic behavior (at $t = 0$) of general solutions to second-order Fuchsian equations using first heuristic arguments, and we determine a "canonical" expansion. Such an expansion involves certain free functions, interpreted as "data on the singularity" and allows us to formulate a singular initial value problem.

The basic understanding of the behavior of Gowdy solutions at $t = 0$ is obtained from the BKL conjecture, and the idea (which we explain in further detail in the second paper [7]) is to neglect spatial derivatives in the evolution equations while solving the remaining ordinary differential equations at each spatial point $x$. This approach allows to identify the leading order terms in the expansion of the solution at $t = 0$. Indeed, the Gowdy equations turn out to be of second-order Fuchsian form, as shown in the second paper. This suggests that we use similar arguments for the derivation of solutions of general second-order Fuchsian equations.

According to the above heuristics, the behavior of general solutions to (2.1) should be driven by the principal part of the PDE's with coefficients evaluated at $t = 0$ if the source-term satisfies certain "decay properties" at $t = 0$, as we will discuss later. More precisely, the two leading terms in the expansion of solutions at $t = 0$ should be determined by the homogeneous equations obtained by setting the right-hand side $f$ in (2.1) to zero. According to the discussion in Section 2.2, any solution $u$ to the second-order Fuchsian equations (2.1) should hence have an

expansion of the form

$$u^{(i)}(t,x) = \begin{cases} u_*^{(i)}(x)\, t^{-a^{(i)}(x)} \ln t + u_{**}^{(i)}(x)\, t^{-a^{(i)}(x)} + O(t^{-a^{(i)}(x)+\alpha^{(i)}}), \\ \qquad\qquad\qquad\qquad\qquad\qquad (a^{(i)}(x))^2 = b^{(i)}(x), \\ \\ u_*^{(i)}(x)\, t^{-\lambda_1^{(i)}(x)} + u_{**}^{(i)}(x)\, t^{-\lambda_2^{(i)}(x)} + O(t^{-\Re\lambda_2^{(i)}(x)+\alpha^{(i)}}), \\ \qquad\qquad\qquad\qquad\qquad\qquad (a^{(i)}(x))^2 \neq b^{(i)}(x), \end{cases} \tag{2.11}$$

for each $i = 1, \dots, n$ and each $x \in U$. Here, the functions $u_*^{(i)}$ and $u_{**}^{(i)}$ are prescribed, and $\alpha^{(i)} > 0$ are real constants. The meaning of the Landau symbols $O$ in this context will be made precise later; at this stage of the discussion they have to be understood "intuitively" as representing terms of higher order in $t$ at $t = 0$. In the case of a transition from the non-degenerate case to the degenerate one or vice versa, the renormalization given by (2.7) is necessary and will always be assumed.

At this stage of the discussion, we clearly see the dependence of the expected leading-order behavior at $t = 0$ on the coefficients of the principal part of the equation. If the roots $\lambda_1$ and $\lambda_2$ are real and distinct, i.e. if $a^2 > b$, we expect a *power-law* behavior. In the degenerate case $\lambda_1 = \lambda_2$, i.e. if $a^2 = b$, we expect a *logarithmic* behavior. Finally, when $\lambda_1$ and $\lambda_2$ are complex for $a^2 < b$, the solution is expected to have an *oscillatory* behavior at $t = 0$ of the form

$$u(t,x) = t^{-a(x)} \big(\tilde u_* \cos(\lambda_I(x) \log t) + \tilde u_{**} \sin(\lambda_I(x) \log t)\big) + \dots$$

for some real coefficient functions $\tilde u_*(x)$ and $\tilde u_{**}(x)$; note that in this case, $\lambda_1 = \bar\lambda_2 = a + i\lambda_I$ with $\lambda_I := \sqrt{b^2 - a}$.

**Relevant function spaces**

Consider any second-order Fuchsian system of the form described in Definition 2.1, with coefficients $a, b, \lambda_1, \lambda_2$ satisfying (2.3). To simplify the presentation, we restrict attention to scalar equations ($n = 1$) and shortly comment on the general case in the course of the discussion.

Fix some integers $l, m \geq 0$ and constants $\alpha, \delta > 0$. For $w \in C^l((0,\delta], H^m(U))$, we define the norm

$$\|w\|_{\delta,\alpha,l,m} := \sup_{0<t\leq\delta} \left( \sum_{p=0}^{l} \sum_{q=0}^{m} \int_U t^{2(\Re\lambda_2(x)-\alpha)} \left|\partial_x^q D^p w(t,x)\right|^2 dx \right)^{1/2}, \tag{2.12}$$

and denote by $X_{\delta,\alpha,l,m}$ the space of all functions with finite norm $\|w\|_{\delta,\alpha,l,m} < \infty$. Throughout, $H^m(U)$ denotes the standard Sobolev space and we recall that all functions are periodic in the variable $x$ with $U$ being a periodicity domain. To cover a system of $n \geq 1$ second-order Fuchsian equations, the norm above is defined by summing over all vector components with different exponents used for different components; recall that each equation in the system will have a different root function $\lambda_2$ and we allow that $\alpha = (\alpha^{(1)}, \dots, \alpha^{(n)})$ is a vector of different positive constants for each equation. The constant $\delta$, however, is assumed to be common for all equations in the system. With this modification, all results in the present section remain valid for systems of equations.

Throughout it is assumed that $\Re\lambda_2$ is continuous and it is then easy to check that $(X_{\delta,\alpha,l,m}, \|\cdot\|_{\delta,\alpha,l,m})$ is a Banach space and that the following property holds.

**Lemma 2.4** (Approximation by smooth functions). *Given any $w \in X_{\delta,\alpha,l,m}$, and constant $\epsilon > 0$, there exists a sequence $(w_n) \in X_{\delta,\alpha,l,m} \cap C^\infty((0,\delta] \times U)$ such that*

$$\lim_{n\to\infty} \|w - w_n\|_{\delta,\alpha-\epsilon,l,m} = 0. \tag{2.13}$$

The functions $w_n$ above are taken to be periodic in space, for instance:

$$w_\eta(t,x) = t^{-\lambda_2(x)} \int_{-\infty}^{\infty} \int_0^{\infty} t^{\lambda_2(y)} w(s,y) k_\eta\Big(\ln \frac{s}{t}\Big) k_\eta(x-y) \frac{1}{s} ds dy. \tag{2.14}$$

Here, $k_\eta : \mathbb{R} \to \mathbb{R}_+$ is any smooth kernel supported in $[-\eta, \eta]$, satisfying $\int_{\mathbb{R}} k_\eta(x) dx = 1$ for all positive $\eta$. We note that the constant $\epsilon > 0$ in (2.13) is introduced in order to guarantee uniform convergence on $(0, \delta]$ in the case where $w$ has no limit at $t = 0$, being just bounded and continuous.

In terms of the spaces $X_{\delta,\alpha,l,m}$, we define in a mathematically precise way a notion of canonical expansions and asymptotic data as follows.

**Definition 2.5.** *Consider a second-order Fuchsian equation* (2.1) *with continuous coefficients* $a, b, \lambda_1, \lambda_2$. *Suppose that* $v$ *and* $w$ *are functions related as follows:*

$$v(t,x) = \begin{cases} u_*(x)\, t^{-a(x)} \ln t + u_{**}(x)\, t^{-a(x)} + w(t,x), & (a(x))^2 = b(x), \\ u_*(x)\, t^{-\lambda_1(x)} + u_{**}(x)\, t^{-\lambda_2(x)} + w(t,x), & (a(x))^2 \neq b(x), \end{cases} \tag{2.15}$$

*for some prescribed data* $u_*$ *and* $u_{**} \in H^{m'}(U)$, *where* $m'$ *is some non-negative integer. Then, one says that* $v$ *satisfies a* ***canonical two-term expansion*** *with* ***asymptotic data*** $u_*$ *and* $u_{**}$ *and* ***remainder*** $w$, *provided* $w \in X_{\delta,\alpha,l,m}$ *for some constants* $\delta, \alpha > 0$ *and non-negative integers* $l, m$.

If the coefficients of the equations are such that there is a continuous transition between the two cases in (2.15), then the asymptotic data functions $u_*$ and $u_{**}$ are renormalized by (2.7).

## 2.4 An existence result for second-order Fuchsian ODE systems

An important property of the ODE solution operator $H$ defined in (2.8) is derived now.

**Proposition 2.6** (Continuity of the ODE solution operator $H$)**.** *Pick up any constants* $\delta > 0$, $\alpha > 0$ *and any integers* $l \geq 1$, $m \geq 0$. *Then, for each* $\epsilon > 0$, *the operator* $H$ *defined in* (2.8) *extends uniquely to a continuous linear map* $X_{\delta,\alpha,l-1,m} \to X_{\delta,\alpha-\epsilon,l,m}$, *and there exists a constant* $C_\epsilon > 0$ *(independent of* $\delta$ *provided* $\delta$ *is sufficiently small), so that*

$$\|H[w]\|_{\delta,\alpha-\epsilon,l,m} \leq C_\epsilon \delta^\epsilon \|w\|_{\delta,\alpha,l-1,m}, \tag{2.16}$$

*for all* $w \in X_{\delta,\alpha,l-1,m}$.

We stress at this stage that for all $l \geq 2$ the extended solution operator $H$ indeed provides the general solution to the Fuchsian equation: given any function $g \in X_{\delta,\alpha,l-1,m}$, the function $w := H[g] \in X_{\delta,\alpha-\epsilon,l,m}$ satisfies the second-order equation

$$D^2 w + 2a\, Dw + b\, w = g,$$

as equality between functions in the space $X_{\delta,\alpha-\epsilon,l-2,m}$.

*Proof of Proposition 2.6.* We restrict attention to the case $a^2 \neq b$, the proof of the other case is completely similar and the transition between the two cases can be understood as a limiting process. Consider first a function $w \in X_{\delta,\alpha,l-1,m} \cap C^\infty((0,\delta] \times U)$. We have

$$\|H[w]\|^2_{\delta,\alpha-\epsilon,l,m} = \sup_{0<t\leq\delta} \int_U t^{2(\Re\lambda_2 - \alpha + \epsilon)} \sum_{q=0}^{m} \sum_{p=0}^{l} |\partial_x^q D^p H[w]|^2\, dx,$$

which is finite thanks to Proposition 2.3 for all $\epsilon > 0$. The term $p = 0$ is expressed explicitly by means of (2.8), while the case $p > 0$ is treated in (2.9) and (2.10). Suppose $(t, x) \in (0, \delta] \times U$. For convenience, we introduce functions $w_1$ and $w_2$ so that

$$\partial_x^q D^p H[w](x) = \int_0^t \partial_x^q (w_2(t,x,s) - w_1(t,x,s)) s^{-1} ds.$$

For $p = 0$, these functions are

$$\begin{aligned} w_1(t,x,s) &:= \frac{1}{\lambda_1(x) - \lambda_2(x)} w(s,x) t^{-\lambda_1(x)} s^{\lambda_1(x)}, \\ w_2(t,x,s) &:= \frac{1}{\lambda_1(x) - \lambda_2(x)} w(s,x) t^{-\lambda_2(x)} s^{\lambda_2(x)}, \end{aligned}$$

while for $p > 0$ we only need to substitute $w$ by $D^{p-1}w$ and complement the expression by factors whose particular form is not relevant for the following. With this, we get

$$\begin{aligned} |\partial_x^q D^p H[w](x)|^2 &= \left| \int_0^t \partial_x^q (w_2(t,x,s) - w_1(t,x,s)) s^{-1} ds \right|^2 \\ &\le 2 \left| \int_0^t \partial_x^q w_2 s^{-1} ds \right|^2 + 2 \left| \int_0^t \partial_x^q w_1 s^{-1} ds \right|^2 \end{aligned}$$

The first term (and in the same way the second one) is handled via Cauchy-Schwarz's inequality (for some constant $\eta > 0$)

$$\begin{aligned} \int_0^t |\partial_x^q w_2 s^{-1}| ds &= \int_0^t |\partial_x^q w_2| s^{-(1+\eta)/2} s^{-(1-\eta)/2} ds \\ &\le \left( \int_0^t |\partial_x^q w_2|^2 s^{-1-\eta} ds \right)^{1/2} \left( \int_0^t s^{-1+\eta} ds \right)^{1/2} \\ &= \left( \frac{1}{\eta} t^\eta \int_0^t |\partial_x^q w_2|^2 s^{-1-\eta} ds \right)^{1/2}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\int_U t^{2(\Re\lambda_2 - \alpha + \epsilon)} |\partial_x^q D^p H[w](x)|^2 dx \\ &\le \frac{2}{\eta} \Big( \int_U \int_0^t t^{2(\Re\lambda_2 - \alpha + \epsilon)} t^\eta (\partial_x^q w_2)^2 s^{-1-\eta} ds dx + \int_U \int_0^t t^{2(\Re\lambda_2 - \alpha + \epsilon)} t^\eta (\partial_x^q w_1)^2 s^{-1-\eta} ds dx \Big). \end{aligned} \tag{2.17}$$

The first term on the right side of this inequality can be written as

$$\begin{aligned} &\int_U \int_0^t t^{2(\Re\lambda_2 - \alpha + \epsilon)} t^\eta (\partial_x^q w_2)^2 s^{-1-\eta} ds dx \\ &= \int_0^t \left( \int_U \left( s^{2(\Re\lambda_2 - \alpha)} (\partial_x^q w_2 t^{\lambda_2} s^{-\lambda_2})^2 s^{2(\epsilon_1 - \eta)} t^{2(\epsilon - \epsilon_1)} \right) \left( \frac{s}{t} \right)^{2(\alpha - \epsilon_1) + 2i\Im\lambda_2} dx \right) t^\eta s^{-1+\eta} ds \qquad (2.18) \\ &\le \frac{1}{\eta} t^{2\eta} \sup_{0<s<t} \left( \int_U \left( s^{2(\Re\lambda_2 - \alpha)} (\partial_x^q w_2 t^{\lambda_2} s^{-\lambda_2})^2 s^{2(\epsilon_1 - \eta)} t^{2(\epsilon - \epsilon_1)} \right) dx \right). \end{aligned}$$

Here we have assumed that the constant $\epsilon_1$ satisfies $\epsilon_1 \leq \alpha$. The significance of the terms $s^{2(\epsilon-\eta)}$ and $t^{2(\epsilon-\epsilon_1)}$ becomes clear in a moment. For the second term on the right side of (2.17), we get similarly

$$
\begin{aligned}
&\int_U \int_0^t t^{2(\Re\lambda_2-\alpha+\epsilon)} t^{\eta} (\partial_x^q w_1)^2 s^{-1-\eta} ds dx \\
&= \int_0^t \left( \int_U \left( s^{2(\Re\lambda_2-\alpha)} (\partial_x^q w_1 t^{\lambda_1} s^{-\lambda_1})^2 s^{2(\epsilon_1-\eta)} t^{2(\epsilon-\epsilon_1)} \right) \left(\frac{s}{t}\right)^{2(\alpha-\epsilon_1)+2(\lambda_1-\lambda_2)+2i\Im\lambda_2} dx \right) t^{\eta} s^{-1+\eta} ds \\
&\leq \frac{1}{\eta} t^{2\eta} \sup_{0<s<t} \left( \int_U \left( s^{2(\Re\lambda_2-\alpha)} (\partial_x^q w_1 t^{\lambda_1} s^{-\lambda_1})^2 s^{2(\epsilon_1-\eta)} t^{2(\epsilon-\epsilon_1)} \right) dx \right),
\end{aligned}
\tag{2.19}
$$

where in the last step we used that $\Re\lambda_1 \geq \Re\lambda_2$. Now, for $p = 0$ we compute,

$$
\begin{aligned}
w_1 t^{\lambda_1} s^{-\lambda_1} &= \frac{1}{\lambda_1-\lambda_2} w(s,x), \\
(\partial_x w_1) t^{\lambda_1} s^{-\lambda_1} &= \partial_x \left( \frac{1}{\lambda_1-\lambda_2} w(s,x) \right) + \left( \frac{1}{\lambda_1-\lambda_2} w(s,x) \right) \partial_x \lambda_1 \ln \frac{t}{s},
\end{aligned}
$$

etc.; analogous formulas hold for $w_2$ and for $p > 0$. So the terms $\partial_x^q w_1 t^{\lambda_1} s^{-\lambda_1}$ incorporate spatial derivatives of $w$ of all order lower or equal to $q$, and all orders lower than $q$ are multiplied with logarithmic terms in $t$ and $s$. Hence, in order to guarantee that the suprema in (2.18) and (2.19) are finite for each $t > 0$, we must choose $\epsilon_1 > \eta$. Moreover, the suprema are uniformly bounded for all $t \in (0, \delta]$, if $\epsilon > \epsilon_1$. With these choices, we can rearrange all terms, introduce a finite constant $C$, which is independent of $\delta$ if $\delta$ is small, as in the hypothesis, and hence obtain

$$
\|H[w]\|_{\delta,\alpha-\epsilon,l,m} \leq C\delta^{\eta} \|w\|_{\delta,\alpha,l-1,m},
$$

for all $w \in X_{\delta,\alpha,l-1,m} \cap C^\infty((0,\delta] \times U)$. Since this inequality holds for all $\eta$ smaller than $\epsilon$, it also holds in the limit $\eta \to \epsilon$. The constant $C$ can hence be adapted so that (2.16) follows for all $w \in X_{\delta,\alpha,l-1,m} \cap C^\infty((0,\delta] \times U)$.

Now let $w$ be a general element in $X_{\delta,\alpha,l-1,m}$ and choose a positive $\epsilon_0$ with $\epsilon_0 < \epsilon$. We set $\widetilde{\epsilon} := \epsilon - \epsilon_0$. According to Lemma 2.4, there exists a sequence $(w_n) \subset X_{\delta,\alpha,l-1,m} \cap C^\infty((0,\delta] \times U)$, so that $\lim_{n\to\infty} \|w - w_n\|_{\delta,\alpha-\epsilon_0,l-1,m} = 0$. Our results for the smooth case show that $(H[w_n])$ is a Cauchy sequence in $X_{\delta,\alpha-\epsilon_0-\widetilde{\epsilon},l-1,m} = X_{\delta,\alpha-\epsilon,l-1,m}$, and we can denote its limit element by $H[w] \in X_{\delta,\alpha-\epsilon,l-1,m}$. In this way, we define the extension of $H$ to the whole space $X_{\delta,\alpha,l-1,m}$. It is then straightforward to see that the limit of the estimate (2.16) leads to the claimed estimate for the full space $X_{\delta,\alpha-\epsilon_0,l-1,m}$. □

Consider a Fuchsian equation (2.1) with right-hand side $f$ of the form (2.2). Let $v$ be a function in the form of Definition 2.5 with remainder $w \in X_{\delta,\alpha,l,m}$ and prescribed data $u_*, u_{**} \in H^{m'}(U)$ for some $\delta > 0$, $\alpha > 0$ and $l, m, m' \geq 0$. Suppose that $m, m'$ are sufficiently large. If necessary, the derivatives in (2.2) are understood in the sense of distributions and we set

$$
F[w](t,x) := f[v](t,x) \tag{2.20}
$$

and finally

$$
G := H \circ F. \tag{2.21}
$$

We are now in a position to define an iteration sequence based on this operator $G$.

**Proposition 2.7** (Iteration sequence)**.** *With the same notation and assumptions as in Proposition 2.6, let $k$ be the number of spatial derivatives in $f$ according to* (2.2) *and $m_0$ another non-negative integer. Suppose that, for given asymptotic data, the operator $F$ satisfies the following regularity assumption (for some $\epsilon_0 > 0$)*

$$F : X_{\delta,\alpha,l,m} \to X_{\delta,\alpha+\epsilon_0,l-1,m-k}, \tag{2.22}$$

*for an integer $l \geq 1$, and all non-negative integers $m$ with $k \leq m \leq m_0$. Let $G$ be the operator defined in* (2.21)*. Then, given any $w_1 \in X_{\delta,\alpha,l,m_0}$, the (in general finite) sequence $(w_j)$ determined by*

$$w_{j+1} = G[w_j], \qquad \textit{for all integers } j \in [1, m_0/k] \tag{2.23}$$

*is well-defined and, moreover, $w_{j+1} \in X_{\delta,\alpha,l,m_0-jk}$.*

Consider a sequence $(w_j)$ defined in the above lemma. It determines a sequence $(v_j)$ for fixed asymptotic data according to (2.15), and all $v_j$ satisfy the canonical two-term expansion. For $l, j \geq 2$, the function $w_j \in X_{\delta,\alpha,l,m}$ satisfies the second-order equation

$$D^2 w_j + 2a\, D w_j + b\, w_j = F[w_{j-1}]$$

as equality in the space $X_{\delta,\alpha,l-2,m}$. The sequence $(w_j)$ has infinitely many elements if $m_0 = \infty$ or if $k = 0$. In the latter case the second-order Fuchsian equation is a system of ordinary differential equations (with the spatial variable $x$ as a parameter). In both cases, a function $w$ is a fixed point of the iteration sequence if and only if the associated function $v$ is a solution of (2.1). A fixed point theorem for the ODE case is as follows.

**Theorem 2.8** (Existence of solutions to second-order Fuchsian ODEs)**.** *Under the assumptions as in Proposition 2.7 with $k = 0$, suppose additionally that for given asymptotic data, the operator $F$ satisfies the following* **Lipschitz continuity property***: for each $r > 0$ and $\epsilon_0 > 0$ arising in* (2.22)*, there exists $\widehat{C} > 0$ independent of $\delta$, so that*

$$\|F[w] - F[\widetilde{w}]\|_{\delta,\alpha+\epsilon_0,l-1,m} \leq \widehat{C}\, \|w - \widetilde{w}\|_{\delta,\alpha,l,m} \tag{2.24}$$

*for all $w, \widetilde{w} \in \overline{B_r(0)} \subset X_{\delta,\alpha,l,m}$. Then, given any initial data $w_1 \in X_{\delta,\alpha,l,m}$ and provided $\delta > 0$ is sufficiently small, the iteration sequence* (2.23) *converges to a unique fixed point $w \in X_{\delta,\alpha,l,m}$.*

*Proof.* Our previous results imply

$$\|G[w] - G[\widetilde{w}]\|_{\delta,\alpha,l,m} \leq \widetilde{C}\delta^\eta \|w - \widetilde{w}\|_{\delta,\alpha,l,m}$$

for a constant $\widetilde{C} > 0$, provided $w, \widetilde{w} \in \overline{B_K}(0) \subset X_{\delta,\alpha,l,m}$. Hence, for sufficiently small $\delta$, the operator $G$ becomes a contraction. The convergence of the iteration sequence follows if we can guarantee that $w_j \in \overline{B_K(0)}$ for a sufficiently large $K$. This, however, is the case since the sequence $(w_j)$ is a Cauchy sequence thanks to the contraction property of $G$ and, hence, is bounded. □

Condition (2.24) guarantees convergence of the sequence $(w_j)$ in the ODE case. The approach of this section is not sufficient to cover the PDE cases of interest and, in general, this condition does not hold if $k > 0$. Moreover, the iteration sequence constructed above is only finite if $m_0$ is finite. We can expect that in typical applications, $m_0$ is infinite if the asymptotic data are smooth and, say, $w_1 = 0$. Still, this does not lead to an existence result except for the analytic case, see [14]. Well-posedness for second-order Fuchsian PDE's in a larger than the analytic class will be addressed in Section 3 after we make the additional assumption that the Fuchsian equations are hyperbolic.

## 2.5 Asymptotic solutions of arbitrary order

The iterative sequence $(w_j)$ has useful asymptotic properties. In order to simplify the discussion in this section, we assume, instead of (2.2), that $f$ has the form

$$f[u](t,x) := f(t,x;u,Du,\partial_x u,\partial_x Du,\partial_x^2 u) \tag{2.25}$$

and that it is a polynomial in all of the arguments involving $u$ with coefficients which are smooth and spatially periodic on $(0,\delta]\times U$. The operator $F$ is defined, in the same way as was done earlier, from given asymptotic data $u_*$ and $u_{**}$. We henceforth assume in the following that $u_*, u_{**} \in H^{m_1}(U)$ for some non-negative integer $m_1$ and that (2.22) holds for $k=2$ and $m_0=m_1$.

**Definition 2.9.** *A function $v$ satisfying the canonical two-term expansion with given asymptotic data is called an* **asymptotic solution of order** $\gamma>0$ *to the system* (2.1) *provided the* ***residual***

$$R[w] := L[w] - F[w]$$

*belongs to $X_{\delta,\gamma+\Re\lambda_2,0,0}$, in which the following notation*

$$L[w] := D^2 w + 2a\,Dw + b\,w, \tag{2.26}$$

*is used for the principal part of* (2.1) *and one assumes that $w \in X_{\delta,\alpha,l,m}$ with $l,m\ge 2$.*

In this definition, we use the obvious generalization of the spaces $X_{\delta,\widetilde{\alpha},l,m}$ to spatially dependent exponents $\widetilde{\alpha}$. Note that $L[w]=L[v]$ if $w$ and $v$ are related as in (2.15). When $l=1$ or $m=1$, one needs to reformulate the operator $L$ (hence $R$) in a weak form, as we will explain below.

**Theorem 2.10.** *Suppose that the operator $F$ satisfies the conditions stated earlier for given asymptotic data, and consider the iteration sequence $w_j \in X_{\delta,\alpha,l,m_1-2(j-1)}$ for $j\ge 2$ given by* (2.23) *with $w_1=0$. Then, for any constant $\kappa<1$ the sequence has the property*

$$w_{j+1}-w_j \in X_{\delta,\alpha+(j-1)\kappa\epsilon_0,l,m_1-2j}$$

*for $1\le j\le m_1/2$. Moreover, the residual satisfies*

$$R[w_j] \in X_{\delta,\alpha+(j-1)\kappa\epsilon_0,l-1,m_1-2(j-2)},$$

*and hence, $w_j$ is an asymptotic solution of order $\gamma=-\Re\lambda_2+\alpha+(j-1)\kappa\epsilon_0$ for $2\le j$.*

This establishes that the order of the asymptotic solution $w_j$ is an increasing function in $j$. Hence, the functions $w_j$ can be interpreted as approximations of actual solutions of increasing accuracy at $t=0$.

*Proof.* In order to show the first relation, we proceed inductively and start with $j=1$. We need to show that $w_2 \in X_{\delta,\alpha,l,m_1-2}$ which is true by Proposition 2.7. Next, we suppose that

$$w_j - w_{j-1} \in X_{\delta,\alpha+(j-2)\kappa\epsilon_0,l,m_1-2(j-1)},$$

has already been shown for a given integer $j\in[2,m_1/2+1]$. We take the difference of the equations for $w_{j+1}$ and $w_j$, and the linearity of the operator $H$ implies

$$w_{j+1}-w_j = H[F[w_j]-F[w_{j-1}]]. \tag{2.27}$$

Since $f$ of the form (2.25) depends smoothly on its arguments by assumption, the mean value theorem implies the existence of a matrix-valued function $M$ of the form

$$M[w_j, w_{j-1}](t,x) := M(t,x; w_j, Dw_j, \partial_x w_j, \partial_x Dw_j, \partial_x^2 w_j, \\ w_{j-1}, Dw_{j-1}, \partial_x w_{j-1}, \partial_x Dw_{j-1}, \partial_x^2 w_{j-1})$$

depending as a polynomial on all arguments involving $w_j$ and $w_{j-1}$ with the property that first

$$M[w_j, w_{j-1}] \in X_{\delta, \Re\lambda_2 - \alpha + \epsilon_0, l-1, m_1 - 2(j-2)},$$

and that second

$$F[w_j] - F[w_{j-1}] = M[w_j, w_{j-1}] \cdot \Delta V[w_j, w_{j-1}]. \tag{2.28}$$

Here, the vector-valued function $\Delta V$ is defined as

$$\Delta V[w_j, w_{j-1}] := (w_j - w_{j-1}, Dw_j - Dw_{j-1}, \partial_x w_j - \partial_x w_{j-1}, \\ \partial_x Dw_j - \partial_x Dw_{j-1}, \ldots, \partial_x^2 w_j - \partial_x^2 w_{j-1})^T.$$

We have,

$$\Delta V[w_j, w_{j-1}] \in X_{\delta, \alpha + (j-2)\kappa\epsilon_0, l-1, m_1 - 2(j-2)}.$$

All this implies that

$$F[w_j] - F[w_{j-1}] \in X_{\delta, \alpha + (j-2)\kappa\epsilon_0 + \epsilon_0, l-1, m_1 - 2(j-2)}.$$

Then, Proposition 2.6 yields

$$w_{j+1} - w_j \in X_{\delta, \alpha + (j-1)\kappa\epsilon_0, l, m_1 - k(j-2)}.$$

At this point we see the significance of the requirement $\kappa < 1$. Namely, we must choose the constant $\epsilon$ in Proposition 2.6 as $\epsilon = \epsilon_0(1-\kappa)$.

The second claim of the theorem is now an immediate consequence. We write the system, which determines $w_j$, in the form

$$L[w_j] - F[w_j] = F[w_{j-1}] - F[w_j].$$

Again, we can write the right side as above. We conclude from the previous results that the right side is in $X_{\delta, \alpha + (j-2)\kappa\epsilon_0 + \epsilon_0, l-1, m_1 - 2(j-2)}$ for $2 \le j$. Since the left side equals $R[w_j]$, the result follows. □

# 3 Second-order hyperbolic Fuchsian systems

## 3.1 Assumptions and basic definitions

From now on we focus on *hyperbolic* second-order Fuchsian equations in the sense of Definition 3.1 below —which form a special case of general second-order Fuchsian equations. Our aim of this section is to establish a well-posedness theory for the (singular) initial value problem when data are prescribed on the singularity.

**Definition 3.1** (Second-order hyperbolic Fuchsian systems). *A second-order hyperbolic Fuchsian system is a set of partial differential equations of the form*

$$D^2 v + 2A\, Dv + B\, v - t^2 K^2 \partial_x^2 v = f[v], \tag{3.1}$$

*in which the function $v : (0, \delta] \times U \to \mathbb{R}^n$ is the main unknown (defined for some $\delta > 0$ and some interval $U$), while the coefficients $A = A(x)$, $B = B(x)$, $K = K(t, x)$ are diagonal $n \times n$ matrix-valued maps and are smooth in $x \in U$ and $t$ in the half-open interval $(0, \delta]$, and $f = f[v](t, x)$ is an $n$-vector-valued map of the following form*

$$f[v](t,x) := f\Big(t, x, v(t,x), Dv(t,x), tK(t,x)\partial_x v(t,x)\Big). \tag{3.2}$$

As earlier, we assume that all functions are periodic with respect to $x$ and that $U$ is a periodicity domain. Further restrictions on the coefficients and on the right-hand side will be imposed and discussed in the course of our investigation. Hence, hyperbolic second-order Fuchsian systems are second-order Fuchsian systems with a particular structure of their right-hand side: in our new notation, we have separated the second-order spatial derivatives from other terms in the right-hand side $f$ and incorporate them into the principal part of the equation, which now reads

$$L := D^2 + 2A\,D + B - t^2 K^2 \partial_x^2. \tag{3.3}$$

This is a linear wave operator for $t > 0$ and, indeed, (3.1) is hyperbolic provided $K$ satisfies (positivity) conditions given below. Later on we will construct solutions where the first three terms of the principal part are of the same order at $t = 0$ and dominant as in the previous section, while the second spatial derivative term is assumed to be of higher order at $t = 0$. Hence, we expect the same phenomenology at $t = 0$ as earlier, and the only significance of the new term in the principal part is that it allows to derive energy estimates.

The eigenvalues of the matrix $K$ are denoted by $k^{(i)}$ and, in the scalar case (or when there is no need to specify the index), we simply write $k$. These quantities are interpreted as characteristic speeds. Throughout this section, we assume that they have the form

$$\begin{aligned} &k^{(i)}(t,x) = t^{\beta^{(i)}(x)} \nu^{(i)}(t,x), \\ &\text{with } \beta^{(i)} : U \to (-1, \infty),\ \nu^{(i)} : [0,\delta] \times U \to (0,\infty) \text{ smooth functions.} \end{aligned} \tag{3.4}$$

In particular, we assume that each derivative of $\nu^{(i)}$ has a finite limit at $t = 0$ for each $x \in U$. The motivation for these assumptions will become clear in the forthcoming discussion. Note we allow for the characteristic speeds to diverge at $t = 0$. At a first glance, this appears to conflict with the standard finite domain of dependence property of hyperbolic equations. Recall that for the standard initial value problem of hyperbolic equations, the solution at a given point is determined by the restriction of the data to a bounded domain of the initial hypersurface; this is a consequence of the finiteness of the characteristic speeds. A closer look at the requirement $\beta(x) > -1$, however, indicates that the characteristic curves are *integrable* at $t = 0$ and, hence that the *finite* domain of dependence property is preserved under our assumptions.

In order to simplify the presentation and without genuine loss of generality, we restrict the discussion now to the scalar case $n = 1$. Consider any second-order Fuchsian hyperbolic equation and for each non-negative integer $l$ and real numbers $\delta, \alpha > 0$, define the space $X_{\delta,\alpha,l}$ by

$$X_{\delta,\alpha,l} := \bigcap_{p=0}^{l} X_{\delta,\alpha,p,l-p},$$

and introduce the norm

$$\|f\|_{\delta,\alpha,l} := \left( \sum_{p=0}^{l} \|f\|^2_{\delta,\alpha,p,l-p} \right)^{1/2}, \qquad f \in X_{\delta,\alpha,l}.$$

Recall that the spaces $X_{\delta,\alpha,l,m}$ and norms $\|\cdot\|_{\delta,\alpha,l,m}$ have been introduced in the previous section. It is straightforward to see that the spaces $(X_{\delta,\alpha,l}, \|\cdot\|_{\delta,\alpha,l})$ have similar properties as the previously defined ones. As we will see in the following discussion, it is not possible to control solutions of our equations in the spaces $X_{\delta,\alpha,l}$ directly. It turns out that we must use spaces $(\tilde{X}_{\delta,\alpha,l}, \|\cdot\|_{\widetilde{\delta,\alpha,l}})$ instead which are defined as earlier, but in the norm $\|f\|_{\widetilde{\delta,\alpha,l}}$ of some function $f$, the highest spatial derivative term $\partial_x^l f$ is weighted with the factor $t^{\beta+1}$. Here $\beta$ is the characteristic speed of the equation given by (3.4). It is easy to see under the earlier conditions that $(\tilde{X}_{\delta,\alpha,l}, \|\cdot\|_{\widetilde{\delta,\alpha,l}})$ are Banach spaces. Moreover, for any $w \in \tilde{X}_{\delta,\alpha,l}$, the mollified function $w_\eta$ defined by (2.14) is an element of $\tilde{X}_{\delta,\alpha-\epsilon,l} \cap C^\infty((0,\delta]\times U)$ for every $\epsilon > 0$. Furthermore, the sequence of mollified functions $w_\eta$ in the limit $\eta \to 0$ converges to $w$ in the norm $\|\cdot\|_{\widetilde{\delta,\alpha-\epsilon,l}}$. We also note that $X_{\delta,\alpha,l} \subset \tilde{X}_{\delta,\alpha,l}$. For our later discussion, let us define

$$X_{\delta,\alpha,\infty} := \bigcap_{l=0}^{\infty} X_{\delta,\alpha,l},$$

and note that $X_{\delta,\alpha,\infty} = \bigcap_{l=0}^{\infty} \tilde{X}_{\delta,\alpha,l}$.

For $w \in \tilde{X}_{\delta,\alpha,1}$, the operator $L$ in (3.3) is defined in the sense of distributions, only, via the following weak form:

$$\begin{aligned}
&\langle \mathcal{L}[w], \phi\rangle \\
&:= \int_0^\delta \int_{\mathbb{R}} t^{\Re\lambda_2(x)-\alpha}\Big( \quad - Dw(t,x)D\phi(t,x) + (2A(x) - \Re\lambda_2(x) + \alpha - 1)\, Dw(t,x)\phi(t,x) \\
&\qquad\qquad + B(x)\, w(t,x)\phi(t,x) + tK(t,x)\partial_x w(t,x) tK(t,x)\partial_x\phi(t,x) \\
&\qquad\qquad + (2t\partial_x K(t,x) + \partial_x\Re\lambda_2(x)K(t,x)t\ln t)tK(t,x)\partial_x w(t,x)\,\phi(t,x)\Big)\, dxdt,
\end{aligned} \tag{3.5}$$

where $\phi$ is any test function, i.e. a real-valued $C^\infty$-function on $(0,\delta]\times\mathbb{R}$ together with some $T \in (0,\delta)$ and a compact set $K \in \mathbb{R}$ so that $\phi(t,x) = 0$ for all $t > T$ and $x \notin K$, and each derivative of $\phi$ has a finite (not necessarily vanishing) limit at $t = 0$ for every $x \in U$. The formula (3.5) is obtained as follows. First we assume that $w$ is a smooth function for $t > 0$ in $\tilde{X}_{\delta,\alpha,1}$. We compute $L[w]$ and multiply (3.3) with $t^{\Re\lambda_2-\alpha}$. Then we integrate this expression in $x$ on $U$ and in $t$ on $[\epsilon,\delta]$ for some $\epsilon > 0$, and integrate by parts. The resulting expression, which resembles (3.5) plus a boundary term at $t = \epsilon$, is meaningful for general $w \in \tilde{X}_{\delta,\alpha,1}$ and only the limit $\epsilon \to 0$ remains to be checked. It turns out that the assumption $w \in \tilde{X}_{\delta,\alpha,1}$ is sufficient to guarantee that this limit is finite and in particular that the boundary term at $t = \epsilon$ resulting from integration by parts goes to zero in the limit. For our later discussion, we note that for any given test function $\phi$, the linear functional $\langle\mathcal{L}[\cdot],\phi\rangle : \tilde{X}_{\delta,\alpha,1} \to \mathbb{R}$ is continuous with respect to the norm $\|\cdot\|_{\widetilde{\delta,\alpha,1}}$. This is the main reason to include the factor $t^{\Re\lambda_2(x)-\alpha}$ in the definition of $\mathcal{L}$.

Consider now functions $u$, $v$, $w$ on $(0,\delta]\times U$ related by

$$v(t,x) = u(t,x) + w(t,x). \tag{3.6}$$

In the following, $v$ will stand for a solution of a Fuchsian system. The function $u$ will be called the leading-order part and $w$ the remainder of the solution at the singularity at $t = 0$. In agreement with the discussion of the canonical two-term expansion in Definition 2.5, we will look for remainders $w$ in spaces $\tilde{X}_{\delta,\alpha,l}$ for some $\alpha > 0$. However, at this stage of the discussion we will not yet fix the particular form of the function $u$ and its dependence on the asymptotic data. Indeed let us assume that $u$ is some given function. In analogy to our earlier discussion, we

introduce the operator $F$ as

$$F[w](t,x) := f[u+w](t,x).$$

If the operator $F$ is a map $\tilde{X}_{\delta,\alpha,1} \to \tilde{X}_{\delta,\alpha,0}, w \mapsto F[w]$, which we shall assume later on, it is meaningful to define its weak form by (for all test functions $\phi$)

$$\langle \mathcal{F}[w], \phi \rangle := \int_0^\delta \int_{\mathbb{R}} t^{\Re\lambda_2(x)-\alpha} F[w](t,x)\phi(t,x)dxdt.$$

**Definition 3.2** (Weak solutions of second-order hyperbolic Fuchsian systems)**.** *Let $u$ be a given function and $\delta, \alpha > 0$ be constants. Then, one says that $w \in \tilde{X}_{\delta,\alpha,1}$ is a weak solution to the second-order hyperbolic Fuchsian equation* (3.8)*, provided*

$$\mathcal{P}[w] := \mathcal{L}[w] + \mathcal{L}[u] - \mathcal{F}[w] = 0. \tag{3.7}$$

For our later discussion, we note that for any given test function $\phi$ and under our earlier assumptions, the linear functional $\langle \mathcal{P}[\cdot], \phi \rangle$ on $\tilde{X}_{\delta,\alpha,1}$ is continuous with respect to the norm $\| \cdot \|_{\tilde{\delta,\alpha,1}}$. For completeness, we also state here that the classical form of the equation for $w$

$$L[w] = F[w] - L[u], \tag{3.8}$$

if $v$ is a classical solution of the equation (3.1). Clearly, if $w$ and $u$ are sufficiently smooth, then $w$ is a weak solution if and only if $w$ is a classical solution of (3.8), or equivalently if $v$ given by (3.6) is a classical solution to the original second-order hyperbolic Fuchsian equation (3.1).

We arrive at the following important notion.

**Definition 3.3** (Singular initial value problem (SIVP))**.** *Consider a second-order hyperbolic Fuchsian equation* (3.8) *with coefficients $(a, b, \lambda_1, \lambda_2)$ and characteristic speeds $k$ satisfying all the conditions stated earlier. Moreover, choose a leading-order part $u$. Then, a function $v$ : $(0, \delta] \times U \to \mathbb{R}$ is called a solution of the* **singular initial value problem** *provided $w := v - u$ belongs to $\tilde{X}_{\delta,\alpha,1}$ for some $\alpha > 0$, and is a weak solution to the second-order hyperbolic Fuchsian system* (3.7).

In particular, we will be interested in the case when $u$ is parametrized by asymptotic data in analogy to the canonical two-term expansion. At this stage of the discussion, however, the particular form of $u$ is not fixed yet.

### 3.2 Linear theory in the space $\tilde{X}_{\delta,\alpha,1}$. Main statement

In this subsection and the following one, we study a particularly fundamental case described by the two conditions:

1. Vanishing leading-order part: $u \equiv 0$.
2. Linear source-term:

   $$F[w](t,x) = f_0(t,x) + f_1(t,x)w + f_2(t,x)Dw + f_3(t,x)tk\partial_x w, \tag{3.9}$$

   with given functions $f_0$, $f_1$, $f_2$, $f_3$, so that $f_1$, $f_2$, $f_3$ are smooth spatially periodic on $(0, \delta] \times U$, and near $t = 0$

   $$\sup_{x \in \bar{U}} f_a(t,x) = O(t^\mu), \qquad a = 1, 2, 3, \tag{3.10}$$

   for some constant $\mu > 0$.

We have not made any assumptions for the function $f_0$ yet, since in the following discussion this function will play a different role than $f_1,\ f_2,\ f_3$. Moreover, no loss of generality is implied by the condition $u \equiv 0$, since the general case can be recovered by absorbing $L[u]$ into the function $f_0$.

Under these assumptions, we pose the question whether there exists a unique weak solution $w$ of the given second-order hyperbolic equation in $\tilde{X}_{\delta,\alpha,1}$ for some $\delta, \alpha > 0$.

**Proposition 3.4** (Existence of solutions of the linear singular initial value problem in $\tilde{X}_{\delta,\alpha,1}$)**.** *Under the assumptions made so far, there exists a unique solution $w \in \tilde{X}_{\delta,\alpha,1}$ of the singular initial value problem for given $\delta, \alpha > 0$ provided:*

1. *The matrix*
$$N := \begin{pmatrix} \Re(\lambda_1-\lambda_2)+\alpha & ((\Im\lambda_1)^2/\eta-\eta)/2 & 0 \\ ((\Im\lambda_1)^2/\eta-\eta)/2 & \alpha & t\partial_x k - \partial_x\Re(\lambda_1-\lambda_2)(tk\ln t) \\ 0 & t\partial_x k - \partial_x\Re(\lambda_1-\lambda_2)(tk\ln t) & \Re(\lambda_1-\lambda_2)+\alpha-1-Dk/k \end{pmatrix} \tag{3.11}$$
*is positive semidefinite at each $(t,x) \in (0,\delta)\times U$ for a constant $\eta > 0$.*

2. *The source-term function $f_0$ is in $X_{\delta,\alpha+\epsilon,0}$ for some $\epsilon > 0$.*

*Then, the solution operator*
$$\mathbb{H} : X_{\delta,\alpha+\epsilon,0} \to \tilde{X}_{\delta,\alpha,1}, \quad f_0 \mapsto w,$$
*is continuous and there exists a finite constant $C_\epsilon > 0$ so that*
$$\|\mathbb{H}[f_0]\|_{\widetilde{\delta,\alpha,1}} \le \delta^\epsilon C_\epsilon \|f_0\|_{\delta,\alpha+\epsilon,0}, \tag{3.12}$$
*for all $f_0$. The constant $C_\epsilon$ can depend on $\delta$, but is bounded for all small $\delta$.*

For reasons that will become clear later on, we call $N$ the **energy dissipation matrix**. We have assumed that $\alpha$ is a positive constant. If, however, $\alpha$ is a positive spatially periodic function in $C^1(U)$, the definition of the spaces $X_{\delta,\alpha,k}$ and $\tilde{X}_{\delta,\alpha,k}$ remains the same, and only the $(2,3)$- and $(3,2)$-components of the energy dissipation matrix $N$ change to $t\partial_x k - \partial_x(\Re(\lambda_1-\lambda_2)+\alpha)(tk\ln t)$. In the following, we continue to assume that $\alpha$ is a constant in order to keep the presentation as simple as possible, but we stress that all following results hold (with this slight change of $N$) if $\alpha$ is a function, and hence no new difficulty arise.

## 3.3 Linear theory in the space $\tilde{X}_{\delta,\alpha,1}$. The proof

The main idea for the proof is to approximate a solution of the *singular* initial value problem by a sequence of solutions of *regular* initial value problems.

**Definition 3.5** (Regular initial value problem (RIVP))**.** *Fix $t_0 \in (0,\delta]$ and some smooth periodic functions $g, h : U \to \mathbb{R}$, and suppose that the right-hand side is of the form* (3.9) *with given smooth spatially periodic functions $f_0,\ f_1,\ f_2,\ f_3$ on $[t_0,\delta]\times U$. Then, $w : [t_0,\delta]\times U \to \mathbb{R}$ is called a solution of the* **regular initial value problem** *associated with the* ***regular data*** *$g, h$ if* (3.8) *holds everywhere on $(t_0,\delta]\times U$ and, moreover,*
$$w(t_0,x) = g(x), \quad \partial_t w(t_0,x) = h(x).$$

For the regular initial value problem, we indeed assume that $f_0$ is smooth just as $f_1$, $f_2$ and $f_3$. By the general theory of linear hyperbolic equations, the regular initial value problem is well-posed, in the sense that there exists a unique smooth solution $w$ defined on $[t_0, \delta]$ for any choice of smooth initial data.

In order to simplify the presentation, we restrict to the scalar case $n = 1$ for this whole section; the general case can be obtained with the same ideas. Choose $\delta, \alpha > 0$ and let $w \in C^1((0, \delta] \times U)$ be a spatially periodic function. Then, we define its **energy** at the time $t \in (0, \delta]$ by

$$\begin{aligned} E[w](t) :=& e^{-\kappa t^\gamma} \int_U t^{2(\lambda_2(x)-\alpha)} \, e[w](t,x) \, dx, \\ e[w](t,x) :=& \frac{1}{2} \left( (\eta \, w(t,x))^2 + (Dw(t,x))^2 + (tk(t,x)\partial_x w(t,x))^2 \right), \end{aligned} \tag{3.13}$$

for some constants $\kappa \geq 0$, $\gamma > 0$ and $\eta > 0$. For convenience, we also introduce the following notation. For any scalar-valued function $w$, we define the vector-valued function

$$\widehat{w}(t,x) := t^{\Re\lambda_2(x)-\alpha}(\eta w(t,x), Dw(t,x), tk(t,x)\partial_x w(t,x)), \tag{3.14}$$

involving the same constants as in the energy. Then, we can write

$$E[w](t) = \frac{1}{2} e^{-\kappa t^\gamma} \|\widehat{w}(t,\cdot)\|^2_{L^2(U)}, \tag{3.15}$$

the norm here being the Euclidean $L^2$-norm for vector-valued functions in $x$. It is important to realize that, provided $\eta > 0$, the expression $\sup_{0<t\leq\delta} \|\widehat{w}(t,\cdot)\|_{L^2(U)}$ for functions of the form (3.14) yields a norm which is equivalent to $\|\cdot\|_{\widetilde{\delta,\alpha,1}}$, thanks to (3.4). Therefore, the energy (3.13) is of relevance for the discussion of functions in $\tilde{X}_{\delta,\alpha,1}$.

**Lemma 3.6** (Fundamental energy estimate for the regular initial value problem)**.** *Suppose that the source-term is of the form* (3.9) *with the conditions* (3.10) *and that the energy dissipation matrix* (3.11) *is positive semidefinite on* $(0, \delta] \times U$ *for given constants* $\alpha, \eta > 0$*. Then, if* $\delta > 0$ *is sufficiently small, there exist constants* $C, \kappa, \gamma > 0$*, independent of the choice of* $t_0 \in (0, \delta]$*, so that for all solutions* $w$ *of the regular initial value problem with smooth regular data at* $t = t_0$*, we have*

$$\begin{aligned} &\|\widehat{w}(t,\cdot)\|_{L^2(U)} \\ &\leq C\, e^{\frac{1}{2}\kappa(t^\gamma - t_0^\gamma)} \Big( \|\widehat{w}(t_0,\cdot)\|_{L^2(U)} + \int_{t_0}^t s^{-1} \|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)} ds \Big), \end{aligned} \tag{3.16}$$

*for all* $t \in [t_0, \delta]$.

The role of the matrix $N$ in (3.11) in the proof of this result motivates the name "energy dissipation matrix". Moreover, this results demonstrates the importance of the assumption $\beta(x) > -1$ in (3.4). Namely, if $\beta(x) \leq -1$ at a point $x \in U$, then for any choice of $\alpha$ and $\eta$, the matrix $N$ would not be positive semidefinite for small $t$ at $x$. While the energy estimate would still be true for a given $t_0$, we would nevertheless lose uniformity of the constants in the estimates with respect to $t_0$. We already stress at this stage that it is this uniformity that will be crucial in the proof of Proposition 3.4.

*Proof of Lemma 3.6.* It will be convenient to work with the function

$$\widetilde{w} := t^\lambda w$$

for some smooth periodic function $\lambda : U \to \mathbb{R}$, in order to optimize the positivity requirement on the matrix $N$ at the end of this proof. Since $w$ is a smooth solution of (3.8),

$$D^2 w + 2aDw + bw - t^2k^2\partial_x^2 w = F[w]$$

for $t \geq t_0$ with coefficients $a$, $b$, (or $\lambda_1$ and $\lambda_2$), and $k$, it follows by direct computation that

$$D^2\widetilde{w} + 2\widetilde{a}D\widetilde{w} + \widetilde{b}\widetilde{w} - t^2k^2\partial_x^2\widetilde{w} = t^\lambda f_0 + F_L[\widetilde{w}].$$

Here, $F_L[\widetilde{w}]$ is an expression linear in $\eta\widetilde{w}$, $D\widetilde{w}$ and $tk\partial_x\widetilde{w}$ with smooth coefficient functions, which are[1] $O(t^{\mu'})$ at $t = 0$ for some constant $\mu' > 0$. Hence, the new source-term is again of the form (3.9) with the conditions (3.10). The coefficients of the principle part are given by

$$\widetilde{a} = a - \lambda, \quad \widetilde{b} = b - 2a\lambda + \lambda^2,$$

so that

$$\widetilde{\lambda}_1 = \lambda_1 - \lambda, \quad \widetilde{\lambda}_2 = \lambda_2 - \lambda.$$

We consider the energy $E[\widetilde{w}]$ with respect to these coefficients and find

$$\begin{aligned}DE[\widetilde{w}] = &- \kappa\gamma t^\gamma E[\widetilde{w}] + e^{-\kappa t^\gamma}\int_U 2(\Re\widetilde{\lambda}_2 - \alpha)t^{2(\Re\widetilde{\lambda}_2-\alpha)}e[\widetilde{w}]dx\\ &+ e^{-\kappa t^\gamma}\int_U t^{2(\Re\widetilde{\lambda}_2-\alpha)}De[\widetilde{w}]dx.\end{aligned}$$

Now,

$$\begin{aligned}De[\widetilde{w}] &= \eta^2\,\widetilde{w}D\widetilde{w} + D\widetilde{w}D^2\widetilde{w} + (1 + Dk/k)(tk\partial_x\widetilde{w})^2 + t^2k^2\partial_x\widetilde{w}\partial_x D\widetilde{w}\\ &= \eta^2\,\widetilde{w}D\widetilde{w} + D\widetilde{w}(-2\widetilde{a}D\widetilde{w} - \widetilde{b}\widetilde{w} + t^2k^2\partial_x^2\widetilde{w} + t^\lambda f_0 + F_L[\widetilde{w}])\\ &\quad + (1 + Dk/k)(tk\partial_x\widetilde{w})^2 + t^2k^2\partial_x\widetilde{w}\partial_x D\widetilde{w}\\ &= -(\widetilde{b} - \eta^2)\widetilde{w}D\widetilde{w} - 2\widetilde{a}(D\widetilde{w})^2 + (1 + Dk/k)(tk\partial_x\widetilde{w})^2 + D\widetilde{w}(t^\lambda f_0 + F_L[\widetilde{w}])\\ &\quad + t^2k^2\partial_x^2\widetilde{w}D\widetilde{w} + t^2k^2\partial_x\widetilde{w}\partial_x D\widetilde{w}.\end{aligned}$$

When $De[\widetilde{w}]$ is multiplied with $t^{2(\Re\widetilde{\lambda}_2(x)-\alpha)}$, the last two terms can be treated as follows

$$\begin{aligned}&t^{2(\Re\widetilde{\lambda}_2(x)-\alpha+1)}(k^2\partial_x^2\widetilde{w}D\widetilde{w} + k^2\partial_x\widetilde{w}\partial_x D\widetilde{w})\\ &= \partial_x(t^{2(\Re\widetilde{\lambda}_2(x)-\alpha+1)}k^2\partial_x\widetilde{w}D\widetilde{w}) - 2t^{2(\Re\widetilde{\lambda}_2(x)-\alpha)}(t\partial_x k)(tk\partial_x\widetilde{w})D\widetilde{w}\\ &\quad - 2t^{2(\Re\widetilde{\lambda}_2(x)-\alpha)}(\partial_x\Re\widetilde{\lambda}_2)(tk\ln t)(tk\partial_x\widetilde{w})D\widetilde{w}.\end{aligned}$$

The first term on the right vanishes after integration in space by virtue of periodicity on the domain $U$. Now we collect all terms of $DE$ as follows

$$DE[\widetilde{w}] =: DE_1[\widetilde{w}] + DE_2[\widetilde{w}],$$

where

$$\begin{aligned}DE_1[\widetilde{w}] := e^{-\kappa t^\gamma}\int_U t^{2(\Re\widetilde{\lambda}_2-\alpha)}\Big(&(\Re\widetilde{\lambda}_2 - \alpha)(\eta\widetilde{w})^2 + (\Re\widetilde{\lambda}_2 - \alpha - 2\widetilde{a})(D\widetilde{w})^2\\ &+ (\Re\widetilde{\lambda}_2 - \alpha + 1 + \frac{Dk}{k})(tk\partial_x\widetilde{w})^2 - (\widetilde{b}/\eta - \eta)(\eta\widetilde{w})D\widetilde{w}\\ &- 2(t\partial_x k + (\partial_x\Re\widetilde{\lambda}_2)(tk\ln t))(tk\partial_x\widetilde{w})D\widetilde{w}\Big)dx,\end{aligned}$$

[1]In the case that $\lambda$ is not a constant, this is strictly speaking only true if $\lambda$ is not too negative.

and

$$DE_2[\widetilde{w}] := \frac{1}{2}e^{-\kappa t^\gamma}\int_U t^{2(\Re\widetilde{\lambda}_2-\alpha)}\Big( -\kappa\gamma t^\gamma(\eta\widetilde{w})^2 - \kappa\gamma t^\gamma(D\widetilde{w})^2 - \kappa\gamma t^\gamma(tk\partial_x\widetilde{w})^2 \\ + 2(t^\lambda f_0 + F_L[\widetilde{w}])D\widetilde{w}\Big)dx.$$

Using the expressions of $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$, we get

$$\begin{aligned}DE_1[\widetilde{w}] := \int_U t^{2(\Re\lambda_2-\lambda-\alpha)}\Big( &-(\lambda-\Re\lambda_2+\alpha)(\eta\widetilde{w})^2 - (\Re\lambda_1-\lambda+\alpha)(D\widetilde{w})^2 \\ &-(\lambda-\Re\lambda_2+\alpha-1-\frac{Dk}{k})(tk\partial_x\widetilde{w})^2 \\ &-\left((\Im\lambda_1)^2/\eta-\eta\right)(\eta\widetilde{w})D\widetilde{w} \\ &-2(t\partial_x k+\partial_x(\Re\lambda_2-\lambda)(tk\ln t))(tk\partial_x\widetilde{w})D\widetilde{w}\Big)dx.\end{aligned}$$

When we choose $\lambda = \Re\lambda_1$, as we will do now, we can write $DE_1$ as follows

$$DE_1[\widetilde{w}] = \int_U -(\widehat{\widetilde{w}}\cdot N\cdot \widehat{\widetilde{w}}^T)dx,$$

where $N$ is the energy dissipation matrix in (3.11) and

$$\widehat{\widetilde{w}} := t^{\Re\widetilde{\lambda}_2(x)-\alpha}(\eta\widetilde{w}(t,x), D\widetilde{w}(t,x), tk(t,x)\partial_x\widetilde{w}(t,x)).$$

Note that $\widehat{\widetilde{w}} = \widehat{w}\cdot T$ with $\widehat{w}$ from (3.14) and

$$T := \begin{pmatrix}1 & \Re\lambda_1/\eta & (\partial_x\Re\lambda_1)(tk\ln t)/\eta \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}. \tag{3.17}$$

The matrix $T$ is invertible and can hence be interpreted as the transformation matrix from the variable $w$ to the variable $\widetilde{w}$ and vice versa. Thus, if $N$ is positive semidefinite at all $(t,x)$, it follows that $DE_1[\widetilde{w}] \le 0$ for all $\widetilde{w}$, and hence for all $w$. According to the hypothesis of this lemma, this is the case, and we are left with

$$DE[\widetilde{w}](t) \le DE_2[\widetilde{w}](t).$$

Denote by $\langle\cdot,\cdot\rangle_{L^2(U)}$ the Euclidean $L^2$-scalar product. Thanks to the properties of the expression $F_L[\widetilde{w}]$, we can choose $\kappa > 0$ (large enough) and $\gamma > 0$ (small enough) so that, uniformly for all $t_0$ (provided $\delta$ is sufficiently small),

$$DE_2[\widetilde{w}](t) \le e^{-\kappa t^\gamma}\langle(0, t^{\Re\lambda_2-\alpha}f_0(t,\cdot), 0), \widehat{\widetilde{w}}(t,\cdot)\rangle_{L^2(U)}.$$

In order to integrate the differential inequality for $E[\widetilde{w}]$ now, we use (3.15) and the Cauchy-Schwarz inequality to obtain

$$D\left(\frac{1}{2}e^{-\kappa t^\gamma}\|\widehat{\widetilde{w}}(t,\cdot)\|^2_{L^2(U)}\right) \le e^{-\kappa t^\gamma}\|t^{\Re\lambda_2-\alpha}f_0(t,\cdot)\|_{L^2(U)}\|\widehat{\widetilde{w}}(t,\cdot)\|_{L^2(U)}.$$

This yields

$$\frac{d}{dt}\|\widehat{\widetilde{w}}(t,\cdot)\|_{L^2(U)} \le \frac{1}{2}\kappa\gamma t^{\gamma-1}\|\widehat{\widetilde{w}}(t,\cdot)\|_{L^2(U)} + t^{-1}\|t^{\Re\lambda_2-\alpha}f_0(t,\cdot)\|_{L^2(U)}.$$

Then the Gronwall inequality implies

$$\|\widehat{\widetilde{w}}(t,\cdot)\|_{L^2(U)} \leq e^{\frac{1}{2}\kappa(t^\gamma - t_0^\gamma)} \left( \|\widehat{\widetilde{w}}(t_0,\cdot)\|_{L^2(U)} + \int_{t_0}^t s^{-1} \|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)} ds \right),$$

for all $t_0 \leq t \leq \delta$. Thanks to the properties of the transformation matrix $T$ defined in (3.17) under our assumptions, one can check that there exists a constant $C > 0$ (independent of $t_0$ and $t$) so that

$$\|\widehat{w}(t,\cdot)\|_{L^2(U)} \leq C e^{\frac{1}{2}\kappa(t^\gamma - t_0^\gamma)} \left( \|\widehat{w}(t_0,\cdot)\|_{L^2(U)} + \int_{t_0}^t s^{-1} \|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)} ds \right).$$

□

*Proof of Proposition 3.4.* We start by assuming a smooth function $f_0$ on $t > 0$, and look for weak solutions $w \in \tilde{X}_{\delta,\alpha,1}$. The first step is to consider a monotonically decreasing sequence $(\tau_n)_{n\in\mathbb{N}} \subset (0,\delta]$ converging to 0. We define a sequence $(w_n)_{n\in\mathbb{N}}$ of functions on $(0,\delta] \times U$ as follows. For all $n \in \mathbb{N}$, we set $w_n(t) = 0$ for all $t \in (0,\tau_n]$. On the interval $[\tau_n,\delta]$, we let $w_n$ be the unique solution of the RIVP for zero regular data at $t_0 = \tau_n$. The linearity of the equation and the conditions on the coefficients imply that $w_n$ is well-defined on the whole interval $(0,\delta]$, and that $w_n \in C^1([0,\delta] \times U)$. It is easy to see that indeed, $(w_n) \subset \tilde{X}_{\delta,\alpha,1}$ for all $\alpha > 0$. The motivation for choosing the sequence $(w_n)$ is that the associated functions $v_n$ (defined according to (3.6)) can be hoped to behave more and more like a solution to the second-order Fuchsian equation obeying the leading-order behavior dictated by $u$, when $n$ tends to infinity; recall that our assumption $u \equiv 0$ represents no loss of generality. Hence, the sequence $(w_n)$ is expected to converge to a solution $w$ of the SIVP. We prove that this is the case making use of the energy estimates for the RIVP derived in Lemma 3.6.

Fix some arbitrary $m, n \in \mathbb{N}$ with $m \geq n$, thus $0 < \tau_m \leq \tau_n \leq \delta$, and define $\xi := w_m - w_n$. Thus, $\xi$ is identically zero on $(0,\tau_m]$, it satisfies (3.8) with the given source-term for $[\tau_m,\tau_n]$, and it satisfies (3.8) with the given source-term but with vanishing inhomogeneity $f_0$ for $[\tau_n,\delta]$ (due to the linearity of the equation). Then, Lemma 3.6 implies that

$$\|\widehat{\xi}(t,\cdot)\|_{L^2(U)} \begin{cases} = 0, & t \in (0,\tau_m], \\ \leq C e^{\frac{1}{2}\kappa(t^\gamma - \tau_m^\gamma)} \int_{\tau_m}^t s^{-1} \|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)} ds, & t \in [\tau_m,\tau_n], \\ \leq C e^{\frac{1}{2}\kappa(t^\gamma - \tau_m^\gamma)} \int_{\tau_m}^{\tau_n} s^{-1} \|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)} ds, & t \in [\tau_n,\delta], \end{cases} \tag{3.18}$$

where $\widehat{\xi}$ is the vector-valued function associated with $\xi$ in the same way as in (3.14). In particular, all constants here are independent of $\tau_m$ and $\tau_n$. We get

$$s^{-\epsilon} \|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)} \leq \|f_0\|_{\delta,\alpha+\epsilon,0}, \tag{3.19}$$

for all $s \in (0,\delta]$. Here, $\epsilon > 0$ is the constant given in the hypothesis of this proposition. In total, the map $s \mapsto s^{-\epsilon} \|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)}$ is a bounded continuous function on $(0,\delta]$ and hence is integrable. Hence, the function

$$G(t) := \int_0^t s^{-1} \|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)} ds$$

is well-defined and finite for all $t$. Indeed, it is continuous for $t \in (0,\delta]$ and $\lim_{t\to 0} G(t) = 0$. This implies that $G$ is uniformly continuous on $(0,\delta]$. By taking the supremum in $t$ on the interval $(0,\delta]$ of (3.18) and adapting the constant $C$ if necessary, we show that

$$\|w_m - w_n\|_{\widetilde{\delta,\alpha,1}} \leq C|G(\tau_n) - G(\tau_m)|. \tag{3.20}$$

In particular, we point out that while $C$ can depend on the choice of $\delta$, it is bounded for small $\delta$. Now, since $G$ is uniformly continuous and $(\tau_n)$ is a Cauchy sequence with limit zero, it follows that $(G(\tau_n))$ is a Cauchy sequence with limit zero. Thus, (3.20) implies that the sequence $(w_n)$ is a Cauchy sequence in $(\tilde{X}_{\delta,\alpha,1}, \|\cdot\|_{\widetilde{\delta,\alpha,1}})$, and hence there exists a limit function $w \in \tilde{X}_{\delta,\alpha,1}$.

Now we have to check that $w$ is a weak solution of the Fuchsian equation, i.e. $\langle \mathcal{P}[w], \phi\rangle = 0$ for all test functions $\phi$ according to (3.7). Consider an arbitrary test-function $\phi$ and pick up any $n \in \mathbb{N}$. The sequence element $w_n$ is constructed so that

$$|\langle \mathcal{P}[w_n], \phi\rangle| \leq \int_0^{\tau_n} \left|\left\langle s^{\Re\lambda_2-\alpha} f_0(s,\cdot), \phi(s,\cdot)\right\rangle_{L^2(U)}\right| ds.$$

This estimate holds since, for all $t > \tau_n$, the approximate solution $w_n$ satisfies the equation (3.8). This yields

$$|\langle \mathcal{P}[w_n], \phi\rangle| \leq \sup_{t\in(0,\delta]} \|t\phi(t,\cdot)\|_{L^2(U)} \int_0^{\tau_n} s^{-1} \left\|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\right\|_{L^2(U)} ds = \widetilde{C}\, G(\tau_n),$$

for some constant $\tilde{C}$. Hence, $\lim_{n\to\infty} \langle \mathcal{P}[w_n], \phi\rangle = 0$. Since $\langle \mathcal{P}[\cdot], \phi\rangle$ is continuous on $\tilde{X}_{\delta,\alpha,1}$ with respect to the norm $\|\cdot\|_{\widetilde{\delta,\alpha,1}}$ for any given test function $\phi$ as noted earlier, it follows that

$$\langle \mathcal{P}[w], \phi\rangle = 0.$$

Hence $w$ is a solution of the SIVP.

Let us check that the limit $w$ of the sequence $(w_n)$ does not depend on the choice of sequence $(\tau_n)$, i.e. that our solution procedure yields a unique solution (which, however, is not guaranteed to be the only solution of the SIVP at this stage of the proof). Let $(\tau_n)$ be a monotonically decreasing sequence in $(0, \delta]$ with limit 0, $(w_n)$ the corresponding sequence of approximate solutions and $w$ its limit; the same for another monotonically decreasing sequence $(\widetilde{\tau}_n)$ in $(0, \delta]$ with limit 0, sequence of approximate solutions $(\widetilde{w}_n)$ and limit $\widetilde{w}$. Now, we take the union of the two sequences $(\tau_n)$ and $(\widetilde{\tau}_n)$ and sort the new sequence $(\widehat{\tau}_n)$, so that it becomes a monotonically decreasing Cauchy sequence with limit 0. Then (3.20) shows that $\|w_n - \widetilde{w}_m\|_{\widetilde{\delta,\alpha,1}} \to 0$ for $n, m \to \infty$. Hence

$$\|w - \widetilde{w}\|_{\widetilde{\delta,\alpha,1}} \leq \|w - w_n\|_{\widetilde{\delta,\alpha,1}} + \|w_n - \widetilde{w}_m\|_{\widetilde{\delta,\alpha,1}} + \|w_m - \widetilde{w}_m\|_{\widetilde{\delta,\alpha,1}} \to 0,$$

and so $w = \widetilde{w}$.

So far, we have shown that for all $f_0 \in X_{\delta,\alpha+\epsilon,0} \cap C^\infty((0,\delta] \times U)$, there exists a weak solution $w \in \tilde{X}_{\delta,\alpha,1}$, and that $w$ is independent of the choice of sequence $(\tau_n)$. Then the solution operator

$$\mathbb{H} : \tilde{X}_{\delta,\alpha+\epsilon,0} \cap C^\infty((0,\delta] \times U) \to \tilde{X}_{\delta,\alpha,1}, \quad f_0 \mapsto w,$$

is well-defined. It is clearly linear, and we derive the estimate (3.12) now. From (3.20), we get

$$\|w\|_{\widetilde{\delta,\alpha,1}} \leq \|w_1\|_{\widetilde{\delta,\alpha,1}} + CG(\delta).$$

We can estimate $\|w_1\|_{\delta,\alpha,1}$ as follows. Because $w_1$ is a solution of the RIVP with zero regular data at $t_0 = \tau_1$, estimate (3.16) yields

$$\|\widehat{w}_1(t,\cdot)\|_{L^2(U)} \leq Ce^{\frac{1}{2}\kappa(t^\gamma-\tau_m^\gamma)} \int_{\tau_1}^t s^{-1}\|s^{\Re\lambda_2-\alpha} f_0(s,\cdot)\|_{L^2(U)} ds$$

for all $t \in [\tau_1, \delta]$. For all $t \in (0, \tau_1]$, we have $\|\widehat{w}_1(t, \cdot)\|_{L^2(U)} = 0$. This shows that $\|w_1\|_{\delta,\alpha,1} \leq CG(\delta)$ with some adapted $C$, and hence, absorbing the factor 2 into the constant, we get

$$\|w\|_{\widetilde{\delta,\alpha,1}} \leq CG(\delta). \tag{3.21}$$

Now, the estimate (3.19) gives

$$G(\delta) \leq \frac{1}{\epsilon}\delta^{\epsilon}\|f_0\|_{\widetilde{\delta,\alpha+\epsilon,0}}.$$

Using this together with (3.21) yields (3.12) on the subset $\tilde{X}_{\delta,\alpha+\epsilon,0} \cap C^{\infty}((0,\delta] \times U)$ of $\tilde{X}_{\delta,\alpha+\epsilon,0}$. Now we proceed in the same way as in the proof of Proposition 2.6 in order to extend this operator and the validity of (3.12) to the full space. For any given test function $\phi$, the expression $\langle \mathcal{P}[w], \phi \rangle$ is continuous with respect to $w$ and $f_0$ (in their respective norms). This is sufficient to show that the extended operator $\mathbb{H}$ maps to a weak solution $w$.

Finally, estimate (3.12) allows us to show uniqueness. Assume that we have two solutions $w$ and $\widetilde{w}$ for the same source-term. Then $w - \widetilde{w}$ is a solution of the same equation with $f_0 = 0$. Hence, (3.12) implies that

$$\|w - \widetilde{w}\|_{\widetilde{\delta,\alpha,1}} \leq 0,$$

and so uniqueness is established. □

## 3.4 Non-linear theory in the spaces $\tilde{X}_{\delta,\alpha,2}$ and $X_{\delta,\alpha,\infty}$

**The general non-linear theory** The well-posedness theory of the previous section, where we restrict to the space $\tilde{X}_{\delta,\alpha,1}$, has certain limitations. First, the statement that the solution of the Fuchsian equation $w$ is an element of $\tilde{X}_{\delta,\alpha,1}$ yields particularly weak information about the behavior of the first spatial derivative at $t = 0$. It would be advantageous if we were able to prove the stronger statement $w \in X_{\delta,\alpha,1}$, possibly under stronger assumptions. Second, it turns out that we need to require a Lipschitz property of the source-term for the general non-linear case which rules out natural non-linearities, for instance quadratic ones, if we only control the first derivatives of the solution. In the case of one spatial dimension, as we always assume in the whole paper, it is sufficient to increase regularity to the space $\tilde{X}_{\delta,\alpha,2}$. It is then clear how to proceed to $\tilde{X}_{\delta,\alpha,k}$ with arbitrary $k \in \mathbb{N}$. Nevertheless, in some applications [7], the space $\tilde{X}_{\delta,\alpha,k}$ imposes too strong a restriction, due to the weak control of the highest spatial derivative. This problem can be avoided by formulating the theory in the space $X_{\delta,\alpha,\infty}$.

**Lemma 3.7** (Existence of solutions of the linear singular initial value problem in $\tilde{X}_{\delta,\alpha,2}$)**.** *Let us make the same assumptions as listed in the beginning of Section 3.2 with $f_a \equiv 0$ for $a = 1, 2, 3$ (for simplicity). Then there exists a unique solution $w \in \tilde{X}_{\delta,\alpha,2}$ of the singular initial value problem for given $\delta, \alpha > 0$ provided:*

1. *The energy dissipation matrix* (3.11) *is positive definite at each $(t, x) \in (0, \delta) \times U$ for a constant $\eta > 0$.*

2. *The source-term function $f_0$ is in $X_{\delta,\alpha+\epsilon,1}$ for some $\epsilon > 0$.*

*Then, the solution operator*

$$\mathbb{H} : X_{\delta,\alpha+\epsilon,1} \to \tilde{X}_{\delta,\alpha,2}, \quad f_0 \mapsto w,$$

*is continuous and there exists a finite constant $C_\epsilon > 0$ so that*

$$\|\mathbb{H}[f_0]\|_{\widetilde{\delta,\alpha,2}} \leq \delta^{\epsilon} C_\epsilon \|f_0\|_{\delta,\alpha+\epsilon,1},$$

*for all $f_0$. The constant $C_\epsilon$ is bounded for all small $\delta$.*

Analogous results hold for systems and for general linear source terms of the form (3.9) with non-vanishing functions $f_1$, $f_2$ and $f_3$ obeying decay conditions analogous to (3.10) also for the first derivatives. Moreover, the result can be generalized to an arbitrary number $k$ of derivatives, i.e. to solutions in the space $\tilde{X}_{\delta,\alpha,k}$. For $k \geq 3$ in one spatial dimension, the Sobolev inequalities imply that the weak derivatives can be identified with classical derivatives. Hence the solution $w \in \tilde{X}_{\delta,\alpha,3}$ of the weak form of the equation (3.7) is then a classical solution of (3.1) with $v = w$, i.e. $u \equiv 0$.

*Proof.* One sees immediately that (the generalization to systems of) Proposition 3.4 applies directly to the system of equations for the unknowns $(w_0, w_1, w_2)$ with $w_0 := w$, $w_1 = Dw$ and $w_2 := t^{\epsilon_1} \partial_x w$, where $\epsilon_1 > 0$ can be any sufficiently small constant. We cannot choose $\epsilon_1 = 0$ since this would lead to a source-term which is not consistent with the hypotheses of Proposition 3.4 in the following. For this system the energy dissipation matrix has the required properties, if it has the required properties for the original equation for $w$ and if we assume the same constant $\alpha$ for the equations for $w_0$, $w_1$ and $w_2$. However, the energy dissipation matrix must positive definite instead of positive semidefinite due to the presence of the non-vanishing constant $\epsilon_1$. One obtains existence and uniqueness in a space $\tilde{X}_{\delta,\alpha,1}$ for vector-valued functions $(w_0, w_1, w_2)$. The thus obtained space $\tilde{X}_{\delta,\alpha,1}$ for vector-valued functions $(w_0, w_1, w_2)$ equals the space $\tilde{X}_{\delta,\tilde{\alpha},2}$ for the original scalar function $w$ where $\tilde{\alpha}$ differs from $\alpha$ by the arbitrarily small constant $\epsilon_1$. ☐

It is important to note that, when we repeat the proof for an arbitrary number of derivatives $k$, the quantity $\tilde{\alpha}$ can be chosen arbitrarily close to $\alpha$ irrespective of the choice of $k$.

**Proposition 3.8** (Existence of solutions of the non-linear singular initial value problem in $\tilde{X}_{\delta,\alpha,2}$)**.** *Suppose that we can choose $\alpha > 0$ so that the energy dissipation matrix* (3.11) *is positive definite at each $(t, x) \in (0, \delta) \times U$ for a constant $\eta > 0$. Suppose that $u \equiv 0$ and that the operator $F$ has the following Lipschitz continuity property: For a constant $\epsilon > 0$ and all sufficiently small $\delta$, the operator $F$ maps $\tilde{X}_{\delta,\alpha,2}$ into $X_{\delta,\alpha+\epsilon,1}$ and, moreover, for each $r > 0$ there exists $\widehat{C} > 0$ (independent of $\delta$) so that*

$$\|F[w] - F[\widetilde{w}]\|_{\delta,\alpha+\epsilon,1} \leq \widehat{C}\, \|w - \widetilde{w}\|_{\widetilde{\delta,\alpha,2}} \tag{3.22}$$

*for all $w, \widetilde{w} \in \overline{B_r(0)} \subset \tilde{X}_{\delta,\alpha,2}$. Then, there exists a unique solution $w \in \tilde{X}_{\delta,\alpha,2}$ of the singular initial value problem.*

The generalization of this result to arbitrarily many derivatives is again straightforward.

*Proof of Proposition 3.8:* Similar to Section 2, we define the operator $\mathbb{G} := \mathbb{H} \circ F$, and argue in the same way as in Proposition 2.7 and Theorem 2.8 that under the hypothesis, this operator is a contraction on closed and bounded subsets of $\tilde{X}_{\delta,\alpha,2}$ if $\delta$ is a sufficiently small. Hence the iteration sequence defined by $w_{j+1} = \mathbb{G}[w_j]$ for $j \geq 1$ and, say, $w_1 = 0$ converges to a fixed point $w \in \tilde{X}_{\delta,\alpha,2}$ with respect to the norm $\| \cdot \|_{\widetilde{\delta,\alpha,2}}$. Because of the properties of $\mathbb{H}$, a fixed point of $\mathbb{G}$ is a solution of the SIVP. Hence, we have shown existence of solutions. Uniqueness can be shown as follows. Given any other solution $\tilde{w}$ in $\tilde{X}_{\delta,\alpha,2}$, it is a fixed point of the iteration $w_{j+1} = \mathbb{G}[w_j]$. Because $\mathbb{G}$ is a contraction, there, however, only exists one fixed point, and hence $\tilde{w} = w$. ☐

**Proposition 3.9** (Existence of solutions of the non-linear singular initial value problem in $X_{\delta,\alpha,\infty}$)**.** *Suppose that we can choose $\alpha > 0$ so that the energy dissipation matrix* (3.11) *is positive definite at each $(t, x) \in (0, \delta) \times U$ for a constant $\eta > 0$. Suppose that $u \equiv 0$ and that the operator $F$ has the following Lipschitz continuity property: For a constant $\epsilon > 0$, every*

*sufficiently small $\delta > 0$ and every non-negative integer $k$, the operator $F$ maps $X_{\delta,\alpha,k+1}$ into $X_{\delta,\alpha+\epsilon,k}$ and, moreover, for each $r > 0$, there exists $\widehat{C} > 0$ (independent of $\delta$) so that*

$$\|F[w] - F[\widetilde{w}]\|_{\delta,\alpha+\epsilon,k} \leq \widehat{C}\, \|w - \widetilde{w}\|_{\widetilde{\delta,\alpha,k+1}} \tag{3.23}$$

*for all $w, \widetilde{w} \in \overline{B_r(0)} \cap X_{\delta,\alpha,k+1} \subset \tilde{X}_{\delta,\alpha,k+1}$. Then, there exists a unique solution $w \in X_{\delta,\alpha,\infty}$ of the singular initial value problem.*

Here, $\overline{B_r(0)}$ is defined with respect to the norm $\|\cdot\|_{\widetilde{\delta,\alpha,k+1}}$. We note that the constant $\widehat{C}$ is allowed to depend on $k$. Note that the Lipschitz estimate involves the norm $\|\cdot\|_{\widetilde{\delta,\alpha,k+1}}$, while the elements for which this estimates needs to be satisfied are required to be only in the subspace $X_{\delta,\alpha,k+1}$ of $\tilde{X}_{\delta,\alpha,k+1}$. The main advantage of this result over the finite differentiability case is that we only need to check that $F$ maps $X_{\delta,\alpha,k+1}$ into $X_{\delta,\alpha+\epsilon,k}$ for all $k$, instead of the stronger statement that $F$ maps $\tilde{X}_{\delta,\alpha,k+1}$ into $X_{\delta,\alpha+\epsilon,k}$ (which would of course, however, only need to hold for finitely many $k$).

*Proof.* We first generalize Lemma 3.7 to the case $f_0 \in X_{\delta,\alpha,\infty}$. Then the solution of the linear equation $\mathbb{H}[f_0] \in X_{\delta,\alpha,\infty}$. Since $F$ maps $X_{\delta,\alpha,\infty}$ to itself, the operator $\mathbb{G} := \mathbb{H} \circ F$ maps $X_{\delta,\alpha,\infty}$ to itself. Hence, the same iteration as in the proof of Proposition 3.8 leads to an iteration sequence $(w_j) \subset X_{\delta,\alpha,\infty}$. Let $k \geq 1$ be arbitrary. According to the hypothesis, $F$ maps $X_{\delta,\alpha,k}$ into $X_{\delta,\alpha+\epsilon,k-1}$. Lemma 3.7 implies that $\mathbb{H}$ maps $X_{\delta,\alpha+\epsilon,k-1}$ to $\tilde{X}_{\delta,\alpha,k}$. Hence $\mathbb{G}$ can be consider as a map $X_{\delta,\alpha,k} \to \tilde{X}_{\delta,\alpha,k}$. We can consider $X_{\delta,\alpha,k}$ to be a subset of $\tilde{X}_{\delta,\alpha,k}$. Then, if we choose $\delta$ small enough, the restriction of $\mathbb{G}$ to $X_{\delta,\alpha,k} \cap \overline{B_r(0)} \subset \tilde{X}_{\delta,\alpha,k}$ is a contraction with respect to the norm $\|\cdot\|_{\widetilde{\delta,\alpha,k}}$ due to (3.23). This implies that for an appropriate choice of $r$, we have that $(w_j)$ is a Cauchy sequence in $X_{\delta,\alpha,\infty} \cap \overline{B_r(0)} \subset \tilde{X}_{\delta,\alpha,k}$ with respect to the norm $\|\cdot\|_{\widetilde{\delta,\alpha,k}}$. Hence, it converges to a limit $w_{(k)} \in \tilde{X}_{\delta(k),\alpha,k}$. We have written $\delta(k)$ now instead of $\delta$ in order to stress that $\delta$ does depend on $k$. We get such a limit function $w_{(k)} \in \tilde{X}_{\delta(k),\alpha,k}$ for all $k \geq 1$. It is straightforward to check that $w_{(k_1)}(t) = w_{(k_2)}(t)$ for all two integers $k_1, k_2 \geq 1$ for all $t \in (0, \min\{\delta(k_1), \delta(k_2)\}]$. Hence, it follows that $w_{(k)} \in X_{\delta(k+1),\alpha,k}$ (without tilde!) for all $k$, and this means that $w_{(k)} \in X_{\delta(k),\alpha,k}$ after possibly decreasing $\delta(k) > 0$ sufficiently for all $k$. So for any given $k$, the limit $w_{(k)}$ is in the range of the operator $\mathbb{G}$, and thus $w_{(k)}$ is the unique fixed point of $\mathbb{G}$ and so the unique solution of the equation in $X_{\delta(k),\alpha,k}$. However, it is not obvious at this point whether we are forced to choose $\delta(k) \to 0$ for $k \to \infty$, and we are left with demonstrating that this is not the case. As soon as we have this, we have constructed the solution $w \in X_{\delta,\alpha,\infty}$ for some $\delta > 0$. This, however, requires only standard arguments for symmetric hyperbolic equations. Hence, we find that we can choose $\delta = \delta(3)$ (for one spatial dimension). ☐

**The (standard) singular initial value problem** The following discussion is devoted to particular choices of the function $u$ motivated by the heuristics introduced in Section 2. Consider the case that $u$ is given by

$$u(t,x) = u_0(t,x) := \begin{cases} u_*(x)\, t^{-a(x)} \ln t + u_{**}(x)\, t^{-a(x)} & a^2 = b, \\ \hat{u}_*(x)\, t^{-\lambda_1(x)} + \hat{u}_{**}(x)\, t^{-\lambda_2(x)}, & a^2 \neq b, \end{cases} \tag{3.24}$$

with $\hat{u}_*, \hat{u}_{**}$ given by (2.7) and with asymptotic data $u_*, u_{**} \in H^3(U)$. In this case, we will speak of the **standard singular initial value problem**[2]. Note that this means that for all $t > 0$, the map $u(t,\cdot)$ and all its time derivatives are in $H^3(U)$.

[2]In order to simplify the language, we often speak of the *singular initial value problem* if there is no risk of confusion.

**Theorem 3.10** (Well-posedness of the standard singular initial value problem in $\tilde{X}_{\delta,\alpha,2}$). *Given arbitrary asymptotic data $u_*, u_{**} \in H^3(U)$, the standard singular initial value problem admits a unique solution $w \in \tilde{X}_{\delta,\alpha,2}$ for $\alpha, \delta > 0$, provided $\delta$ is sufficiently small and the following conditions hold:*

1. *Positivity condition. Suppose that we can choose $\alpha > 0$ so that the energy dissipation matrix* (3.11) *is positive definite at each $(t, x) \in (0, \delta) \times U$ for a constant $\eta > 0$.*
2. *Lipschitz continuity property. For the given $\alpha > 0$, the operator $F$ satisfies the Lipschitz continuity property stated in Proposition 3.8 for all asymptotic data $u_*, u_{**} \in H^3(U)$ for some $\epsilon > 0$.*
3. *Integrability condition. The constants $\alpha$ and $\epsilon$ satisfy*

$$\alpha + \epsilon < 2(\beta(x) + 1) - \Re(\lambda_1(x) - \lambda_2(x)), \qquad x \in U. \tag{3.25}$$

An analogous theorem can be formulated for the $C^\infty$-case based on Proposition 3.9. In this case, the asymptotic data $u_*$, $u_{**}$ must be in $C^\infty(U)$ and the Lipschitz condition must be substituted by the condition of Proposition 3.9. The unique solution $w$ of the singular initial value problem is then an element of $X_{\delta,\alpha,\infty}$.

We note that there might be room for improvements in the finite differentiability case $k = 2$, since three derivatives of the asymptotic data yield control of only two derivatives of the solution.

*Proof.* We can apply Proposition 3.8 if we are able to control the additional contribution of the term $L[u]$ which has to be considered as part of the source-term. It has no contribution to the Lipschitz estimate (3.22), but we have to guarantee that under these hypotheses, $L[u] \in X_{\delta,\alpha+\epsilon,1}$ for the given constant $\epsilon$. This is indeed the case if (3.25) holds. ☐

**Example 3.11.** *Consider the second-order hyperbolic Fuchsian equation*

$$D^2 v - \lambda D v - t^2 \partial_x^2 v = 0,$$

*with a constant $\lambda$. This is the Euler-Poisson-Darboux equation. In the standard notation it is*

$$\partial_t^2 v - \partial_x^2 v = \frac{1}{t}(\lambda - 1)\partial_t v.$$

*Note that $\lambda = 1$ is the standard wave equation, and in this case, the standard singular initial value problem reduces to the standard Cauchy problem.*

1. *Case $\lambda \geq 0$. With our notation, we have $\lambda_1 = 0$, $\lambda_2 = -\lambda$, $\beta \equiv 0$, $\nu \equiv 1$ and $f \equiv 0$. The positivity condition of the energy dissipation matrix* (3.11) *is satisfied precisely for $\alpha \geq 1 - \lambda$ and all sufficiently small $\eta > 0$. The integrability condition* (3.25) *is satisfied precisely for $\lambda < 2 - \alpha$. Hence, our previous proposition implies that the singular initial value problem is well-posed, provided*

$$0 \leq \lambda < 2. \tag{3.26}$$

*Namely, in this case there exists a solution $w$ in $\tilde{X}_{\delta,\alpha,2}$ for some $\alpha > 0$ for arbitrary asymptotic data in $H^3(U)$.*

2. *Case $\lambda < 0$. With our notation, we have $\lambda_1 = |\lambda|$, $\lambda_2 = 0$, $\beta \equiv 0$, $\nu \equiv 1$ and $f \equiv 0$. The positivity condition of the energy dissipation matrix* (3.11) *is satisfied precisely for $\alpha \geq 1 - |\lambda|$ and all sufficiently small $\eta > 0$. The integrability condition* (3.25) *is satisfied precisely for $|\lambda| < 2 - \alpha$. Hence, our previous proposition implies that the singular initial value problem is well-posed, provided*

$$-2 < \lambda < 0.$$

*Namely, in this case there exists a solution $w$ in $\tilde{X}_{\delta,\alpha,2}$ for some $\alpha > 0$ for arbitrary asymptotic data in $H^3(U)$.*

*Now, it turns out that general smooth solutions to the Euler-Poisson-Darboux equation can be expressed explicitly by a Fourier ansatz in $x$ and by Bessel functions in $t$. It is then easy to check that* (3.26) *(and similarly for $\lambda < 0$) is sharp: While for $0 \leq \lambda < 2$, all solutions of the equation behave consistently with the two-term expansion at $t = 0$, this is not the case for $\lambda \geq 2$ for general asymptotic data. Hence the singular initial value problem is not well-posed for $\lambda \geq 2$. This is completely consistent with our heuristic discussion in Section 2.3. Namely, if $\lambda = 2$, the assumption that the source-term $t^2\partial_x^2 v$ is negligible at $t = 0$ fails since it is of the same order in $t$ at $t = 0$ as the second leading-order term. However, we can see in the proof of Theorem 3.10 that in the special case $u_* = 0$ (and arbitrary $u_{**}$), the integrability condition* (3.25) *can be relaxed. For this special choice of data, solutions to the singular initial value problem exist even for $\lambda \geq 2$.*

**Singular singular initial value problems with asymptotic solutions of order** $j$ One of the main aims of this paper is to study the well-posedness of the standard singular initial value problem just discussed. In this sense, we can be satisfied with Theorem 3.10. However, it turns out that, often in applications, the three conditions in this theorem cannot be satisfied simultaneously. While it is often possible to find constants $\alpha$ and $\epsilon$ in accordance with the second and third condition, it can turn out that the corresponding choice of $\alpha$ is too small to make the energy dissipation matrix positive definite. The following trick can sometimes solve this problem.

For the following discussion, we need to bring together results from Sections 2 and 3, and we are forced to distinguish between operators now which for the sake of simplicity have carried the same name so far. Consider some asymptotic data $u_*$ and $u_{**}$ and define the function $u_0$ as in (3.24). We write $\hat{f}$ for the source-term in (2.1) and continue to write $f$ for the source-term in (3.1). Accordingly, we write $\hat{F}[w] := \hat{f}[u_0 + w]$ and $F[w] := f[u_0 + w]$, so that e.g.

$$\hat{F}[w] = F[w] + t^2 k^2 \partial_x^2 (u_0 + w).$$

In the same way, we distinguish between operators $L$ and $\hat{L}$. Now we make the same assumptions on $\hat{F}$ as in Proposition 2.7 in Section 2.5. If the asymptotic data is in $H^{m_1}(U)$ for some positive integer $m_1$, then the function $u_j$, referred to as $w_j$ in Proposition 2.7, with $u_1 = 0$ is well-defined in $X_{\delta,\tilde{\alpha},l,m_1-2(j-1)}$ for some $\tilde{\alpha} > 0$ and all $j$ with $m_1/2 + 1 \geq j \geq 1$, provided $F$ maps $X_{\delta,\tilde{\alpha},l,m}$ to $X_{\delta,\tilde{\alpha}+\epsilon,l-1,m-1}$ for all $m \leq m_1$, for some integer $l \geq 1$ and for some $\epsilon > 0$.

Now let us choose the leading-order function $u$ as

$$u(t, x) = u_0(t, x) + u_j(t, x), \tag{3.27}$$

for $j$ in the range given above. We refer to the singular initial value problem based on this choice of leading-order term as **singular initial value problem with asymptotic solutions of order** $j$. For $j = 1$, it reduces to the standard singular initial value problem; hence we will focus on the case $j \geq 2$ in the following. Note that, if $w$ is a solution of the singular initial value problem of order $j$, it is also a solution of the standard singular initial value problem. However,

if there is only one solution $w$ of the singular initial value problem with asymptotic solutions of order $j$ for given asymptotic data, it does not mean that $w$ is the only solution of the standard initial value problem for the same asymptotic data.

It can be seen easily that the remainder $w$ of a solution of the singular initial value problem with asymptotic solutions of order $j$ satisfies the equation

$$L[w] = F_j[w] := F[u_j + w] - F[u_{j-1}] + t^2 k^2 \partial_x^2 (u_j - u_{j-1})$$

for $j \geq 2$. Now thanks to Theorem 2.10 we have

$$u_j - u_{j-1} \in X_{\delta, \tilde{\alpha}+(j-2)\kappa\epsilon_0, l, m_1 - 2(j-1)},$$

for all $\kappa < 1$. Hence it is reasonable to restrict to remainders $w \in X_{\delta,\tilde{\alpha}+(j-2)\kappa\epsilon_0,l,m_1-2(j-1)-1}$ in the following. One finds easily that this means that $F_j[w] \in X_{\delta,\tilde{\alpha}+(j-1)\kappa\epsilon_0,l-1,m_1-2(j-1)-2}$ if $2(\beta(x)+1) > \kappa\epsilon$ for all $x \in U$ by using similar arguments as in Theorem 2.10. This gives us hope that we can apply Proposition 3.8 with

$$\alpha := \tilde{\alpha} + (j-2)\kappa\epsilon. \tag{3.28}$$

The effect of the ansatz (3.27) is a value of $\alpha$ which increases $\tilde{\alpha}$ by $(j-2)\kappa\epsilon$. Namely if $m_1$ is sufficiently large, we can choose $j$ large enough so that the energy dissipation matrix, evaluated for $\alpha$, can become positive definite. The main prize that we pay with this approach is that the asymptotic data must be sufficiently regular and that we must live with a loss of regularity which is stronger the larger $j$ is.

For the statement of the following theorem, we need the following notation. For all $w \in X_{\delta,\alpha,k}$ (or $w \in \tilde{X}_{\delta,\alpha,k}$ respectively), we introduce the functions $E_{\delta,\alpha,k}[w] : (0,\delta] \to \mathbb{R}$ (or $\tilde{E}_{\delta,\alpha,k}[w] : (0,\delta] \to \mathbb{R}$ respectively) which are defined in the same way as the respective norms, but the supremum in $t$ has not been evaluated yet. In particular, this means that $E_{\delta,\alpha,k}[w]$ (or $\tilde{E}_{\delta,\alpha,k}[w]$) is a bounded continuous function on $(0,\delta]$.

**Theorem 3.12** (Well-posedness of the singular initial value problem with asymptotic solutions of higher-order in $\tilde{X}_{\delta,\alpha,2}$)**.** *Given any integer $j \geq 2$ and any asymptotic data $u_*, u_{**} \in H^{m_1}(U)$ with $m_1 = 2j+1$, there exists a unique solution $w \in \tilde{X}_{\delta,\alpha,2}$ of the singular initial value problem with asymptotic solutions of order $j$ for some $\alpha > 0$, provided*

1. *$F$ maps $\tilde{X}_{\delta,\tilde{\alpha},m_1}$ into $X_{\delta,\tilde{\alpha}+\epsilon,m_1-1}$ for all asymptotic data $u_*, u_{**} \in H^{m_1}(U)$ for some $\epsilon > 0$ and $\tilde{\alpha}$ given by (3.28) for an arbitrary $\kappa < 1$.*

2. *The characteristic speed satisfies*

$$2(\beta(x)+1) > \kappa\epsilon \quad \text{for all } x \in U$$

   *for the same constant $\kappa$ chosen earlier.*

3. *$F$ satisfies the following Lipschitz condition: for each $r > 0$ there exists a constant $C > 0$ (independent of $\delta$) so that*

$$E_{\delta,\tilde{\alpha}+\epsilon,1}[F[w] - F[\widetilde{w}]](t) \leq C\tilde{E}_{\delta,\tilde{\alpha},2}[w - \widetilde{w}](t)$$

   *for all $t \in (0,\delta]$ and for all $w, \widetilde{w} \in \overline{B_r(0)} \subset \tilde{X}_{\delta,\tilde{\alpha},2}$.*

4. *The energy dissipation matrix (3.11) (evaluated with $\alpha$) is positive definite at each $(t,x) \in (0,\delta) \times U$ for a constant $\eta > 0$.*

The third condition above is meaningful since both sides of the inequality are continuous and bounded functions on $(0, \delta]$. Note that this theorem can be formulated without difficulty for the $C^\infty$-case and leads to a simpler statement.

*Proof.* Under the first hypothesis, it can be shown similar to Proposition 2.7 that the $j$-th element of the iteration sequence $u_j$ is in $X_{\delta,\tilde{\alpha},m_1-2(j-1)}$. As in Theorem 2.10 it can be demonstrated that $u_j - u_{j-1}$ is in $X_{\delta,\tilde{\alpha}+(j-2)\kappa\epsilon,m_1-2(j-1)}$ for all $j \geq 2$. Now let $w \in \tilde{X}_{\delta,\tilde{\alpha}+(j-2)\kappa\epsilon,m_1-2(j-1)-1}$. Under the second hypothesis, it follows that $F_j[w] \in X_{\delta,\tilde{\alpha}+j\kappa\epsilon,m_1-2(j-1)-2}$. Now, in order to apply Proposition 3.8 with $F$ substituted by $F_j$, it is necessary to choose $m_1 = 2j + 1$. The operator $F_j$ satisfies the Lipschitz condition of Proposition 3.8 if the third hypothesis is satisfied. Thanks to the fourth assumption, we can now apply Proposition 3.8. □

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